

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Thursday, August 30

<http://math.arizona.edu/~cushing/250a.html>

GRAPHS OF EXPONENTIALS

Sketch the graph of $P(x) = e^x$

Notes

(1) $\frac{dP}{dx} = e^x > 0$ implies graph is *increasing*

(2) $\frac{d^2P}{dx^2} = e^x > 0$ implies graph is *concave up*

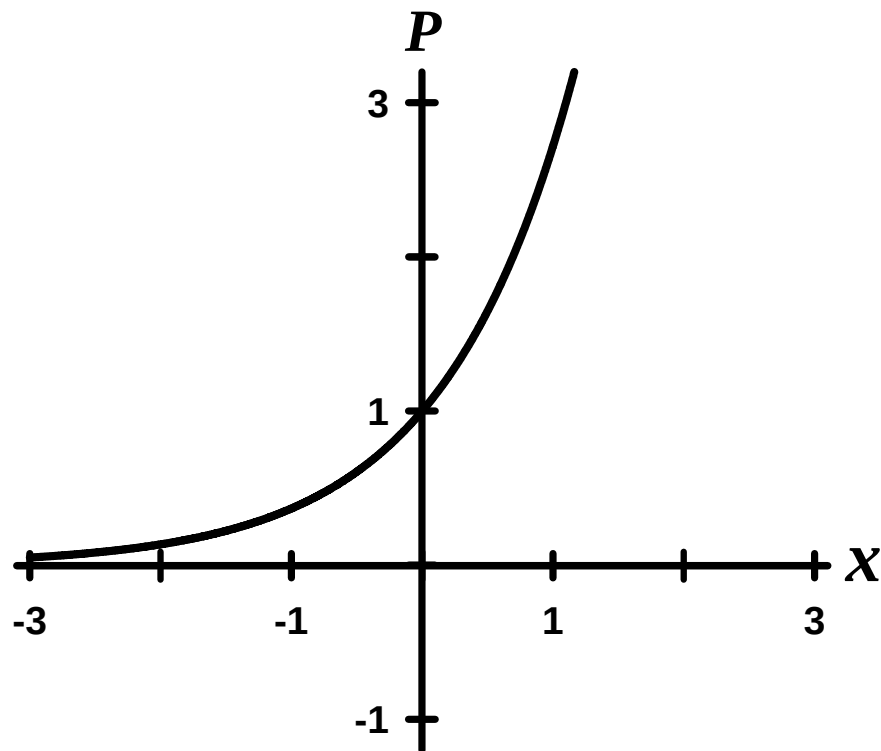
(3) $P(x) > 0$ for all x (graph lies above the horizontal x -axis)

(4) $P(0) = e^0 = 1$ (graph passes through point $(0,1)$)

(5) $\lim_{x \rightarrow +\infty} P(x) = +\infty, \quad \lim_{x \rightarrow -\infty} P(x) = 0$

GRAPHS OF EXPONENTIALS

Sketch the graph of $P(x) = e^x$



x -axis is called a “horizontal asymptote”

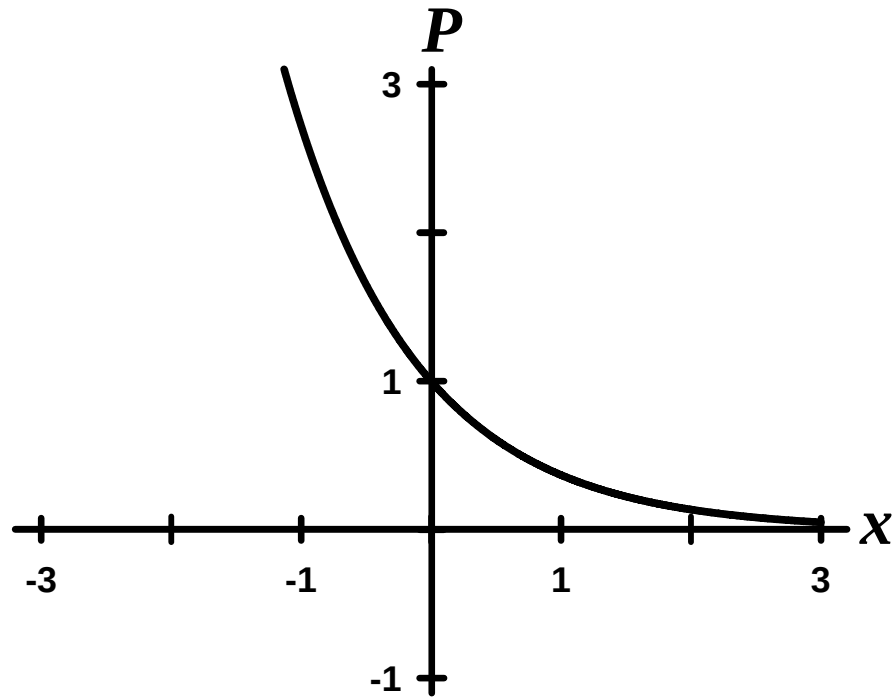
GRAPHS OF EXPONENTIALS

Sketch the graph of $P(x) = e^{-x}$

Reflect graph of e^x through the vertical axis

GRAPHS OF EXPONENTIALS

Sketch the graph of $P(x) = e^{-x}$



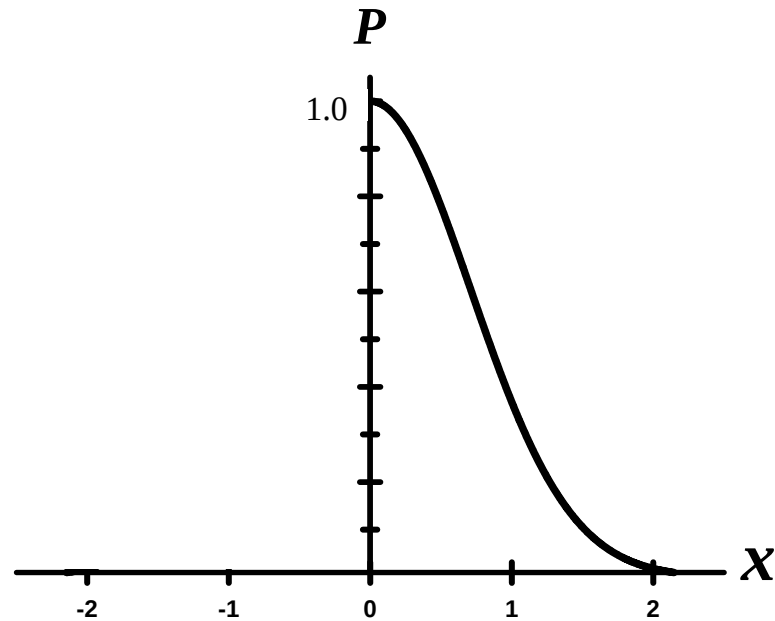
GRAPHS OF EXPONENTIALS

Sketch the graph of $P(x) = e^{-x^2}$

(1) $\frac{dP}{dx} = -2xe^{-x^2} < 0$ for $x > 0 \Rightarrow$ implies graph is *decreasing* for $x > 0$

(2) $\frac{d^2P}{dx^2} = 2(2x^2 - 1)e^{-x^2} \Rightarrow \begin{cases} \text{concave down for } 0 < x < \sqrt{2}/2 \\ \text{concave up for } \sqrt{2}/2 < x \end{cases}$

(3) $P(x) > 0$, $P(0) = 1$, $\lim_{x \rightarrow +\infty} P(x) = 0$



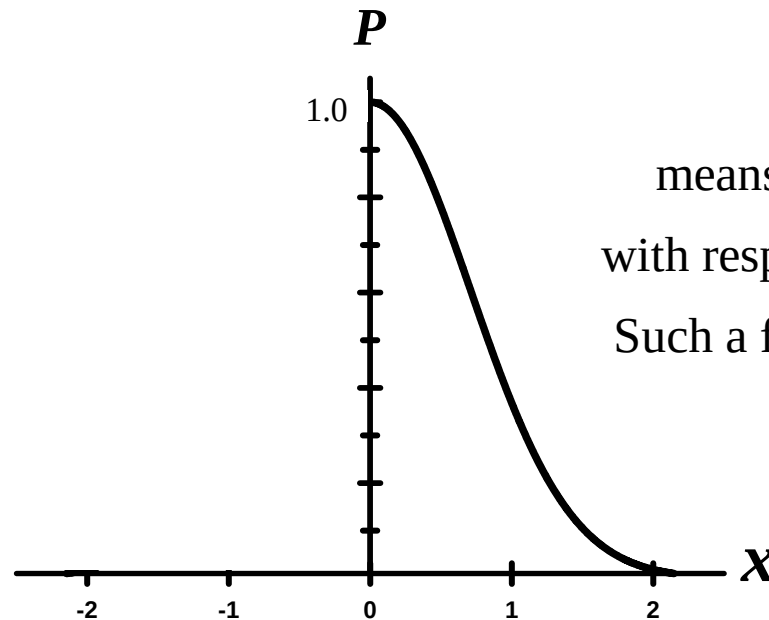
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(3) $P(x) > 0$, $P(0) = 1$, $\lim_{x \rightarrow +\infty} P(x) = 0$



$$P(-x) = P(x)$$

means graph is symmetric
with respect to the vertical axis.
Such a function is called *even*.

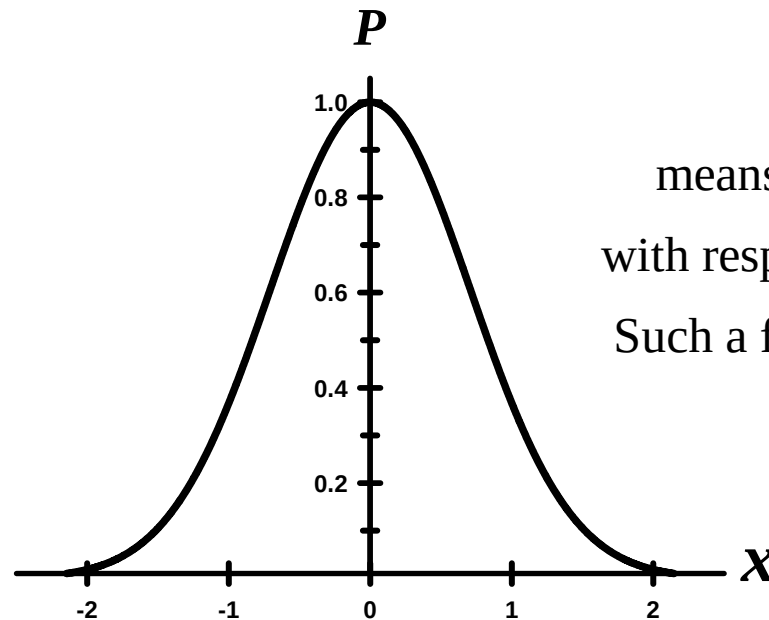
GRAPHS OF EXPONENTIALS

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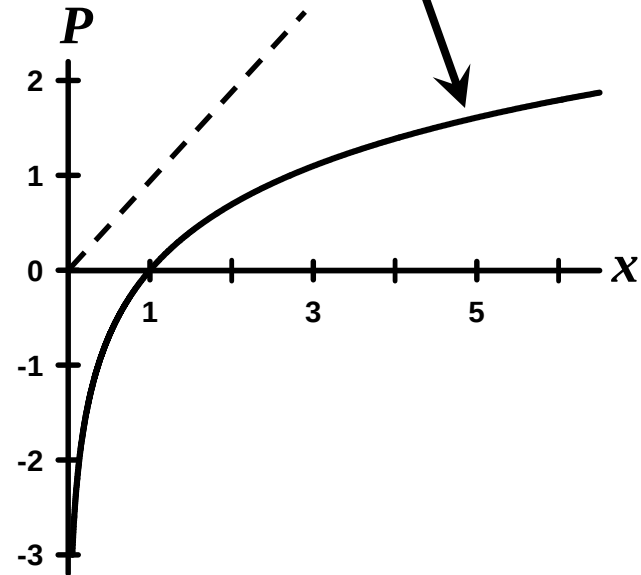
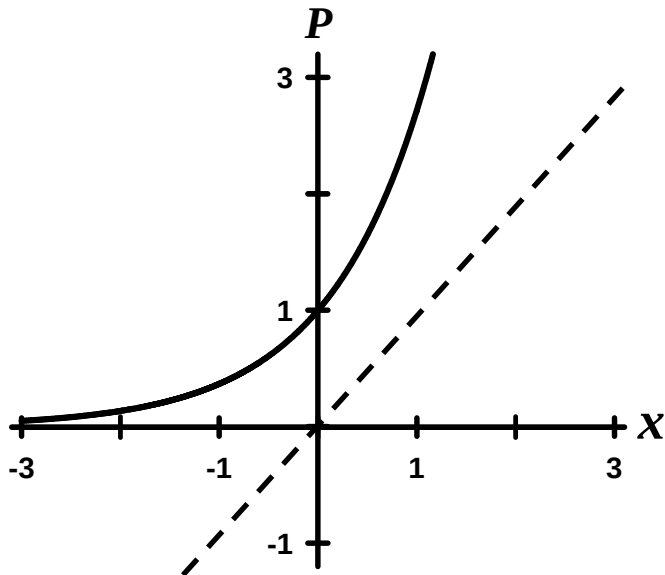
$$P(-x) = P(x)$$

means graph is symmetric
with respect to the vertical axis.
Such a function is called *even*.

GRAPHS OF LOGARITHMIC FUNCTIONS

Sketch the graph of $P(x) = \ln(x)$

Inverse of $P(x) = e^x$



y -axis is a vertical asymptote

GRAPHS OF LOGARITHMIC FUNCTIONS

Sketch the graph of $P(x) = \ln(x)$

Notes

$$\frac{dP}{dx} = \frac{1}{x} > 0 \text{ for } x > 0 \text{ implies graph is } \textit{increasing}$$

$$\frac{d^2P}{dx^2} = -\frac{1}{x^2} < 0 \text{ implies graph is } \textit{concave down}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

How long does it take a decreasing exponential function $P(t)$ that starts at P_0 at time $t = 0$ to decrease by 50% ?

How does this halving time t_h depend on P_0 ?

How does t_h depend on the exponential rate of decay ?

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

$$P(t) = P_0 e^{-rt}$$

$$0 < r = \text{decay rate}$$

For what value of $t = t_h$ does $P(t_h) = \frac{1}{2} P_0$?

$$\text{Solve } P_0 e^{-rt_h} = \frac{1}{2} P_0 \text{ for } t_h.$$

First cancel P_0 .

$$e^{-rt_h} = \frac{1}{2} \Rightarrow t_h = -\frac{1}{r} \ln \frac{1}{2}$$

$$\text{Halving time: } t_h = \frac{1}{r} \ln 2$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

How long does it take a decreasing exponential function $P(t)$ that starts at P_0 at time $t = 0$ to decrease by 50% ?

$$\text{ANSWER: } t_h = \frac{1}{r} \ln 2$$

where r is the exponential decay rate.

How does this halving time t_h depend on P_0 ?

ANSWER: it doesn't depend on P_0 at all !

How does t_h depend on the exponential rate of decay ?

ANSWER: it is *inversely proportional* to r .

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

How long does it take a decreasing exponential function $P(t)$ that starts at P_0 at time $t = 0$ to decrease by 50% ?

$$\text{ANSWER: } t_h = \frac{1}{r} \ln 2$$

where r is the exponential decay rate.

t_h is called the *half life* of P .

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

EXAMPLE

Suppose that the amount P of a drug of a medicinal drug present in a patient's bloodstream decreases by $p\%$ per hour.

- (a) Show that P is a decreasing exponential function and find its half-life.
- (b) For the antibiotic ampicillis $p = 40\%$. What is the half-life? Draw a graph of P as a function of time.

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

EXAMPLE

Suppose that the amount P of a drug of a medicinal drug present in a patient's bloodstream decreases by $p\%$ per hour.

(a) Show that P is a decreasing exponential function and find its half-life.

$$\begin{aligned} P(1) = P(0) \left(1 - \frac{p}{100} \right) &\Rightarrow P(2) = P(1) \left(1 - \frac{p}{100} \right) = P(0) \left(1 - \frac{p}{100} \right)^2 \\ &\Rightarrow P(3) = P(2) \left(1 - \frac{p}{100} \right) = P(0) \left(1 - \frac{p}{100} \right)^3 \end{aligned}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

EXAMPLE

Suppose that the amount P of a drug of a medicinal drug present in a patient's bloodstream decreases by $p\%$ per hour.

- (a) Show that P is a decreasing exponential function and find its half-life.

$$P(1) = P(0) \left(1 - \frac{p}{100} \right)$$

Observe the pattern ... and induct :

$$P(2) = P(0) \left(1 - \frac{p}{100} \right)^2$$

$$P(t) = P(0) \left(1 - \frac{p}{100} \right)^t$$

This is an exponential function.

$$P(3) = P(0) \left(1 - \frac{p}{100} \right)^3$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

EXAMPLE

Suppose that the amount P of a drug of a medicinal drug present in a patient's bloodstream decreases by $p\%$ per hour.

- (a) Show that P is a decreasing exponential function and find its half-life.

$$P(t) = P(0)a^t \quad \text{where} \quad a = 1 - \frac{p}{100} < 1$$

$$P(t) = P(0)e^{(\ln a)t}$$

$$P(t) = P(0)e^{-rt} \quad \text{where} \quad r = -\ln a > 0$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

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$$P(t) = P(0)e^{-rt} \quad \text{where} \quad r = -\ln a > 0$$

$$\text{Half life: } t_h = \frac{\ln 2}{r} = \frac{2}{-\ln a} = \frac{2}{-\ln(1 - p/100)}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

EXAMPLE

Suppose that the amount P of a drug of a medicinal drug present in a patient's bloodstream decreases by $p\%$ per hour.

(b) For the antibiotic ampicillin $p = 40\%$. What is the half-life?

Draw a graph of P as a function of time.

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(1) Exponential Decay

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(b) For the antibiotic ampicillis $p = 40\%$. What is the half-life?

Draw a graph of P as a function of time.

$$\text{Half life: } t_h = \frac{2}{-\ln(1 - 40/100)} \approx 1.37 \text{ hour}$$

$$P(t) = P(0)e^{-rt} \text{ where } r \approx 0.51$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

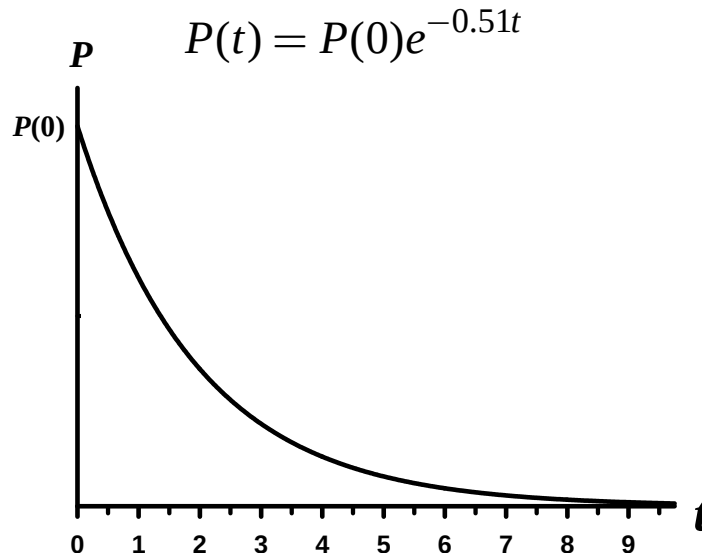
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APPLICATIONS OF EXPONENTIAL FUNCTIONS

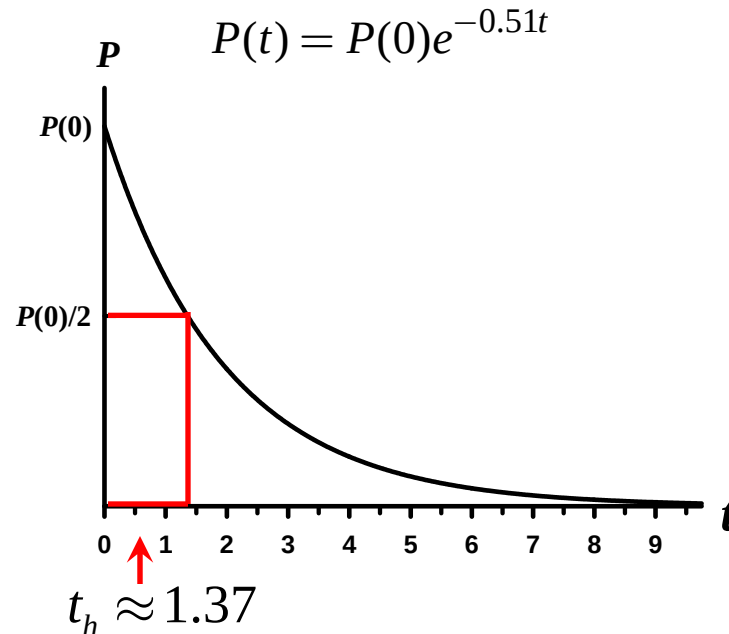
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APPLICATIONS OF EXPONENTIAL FUNCTIONS

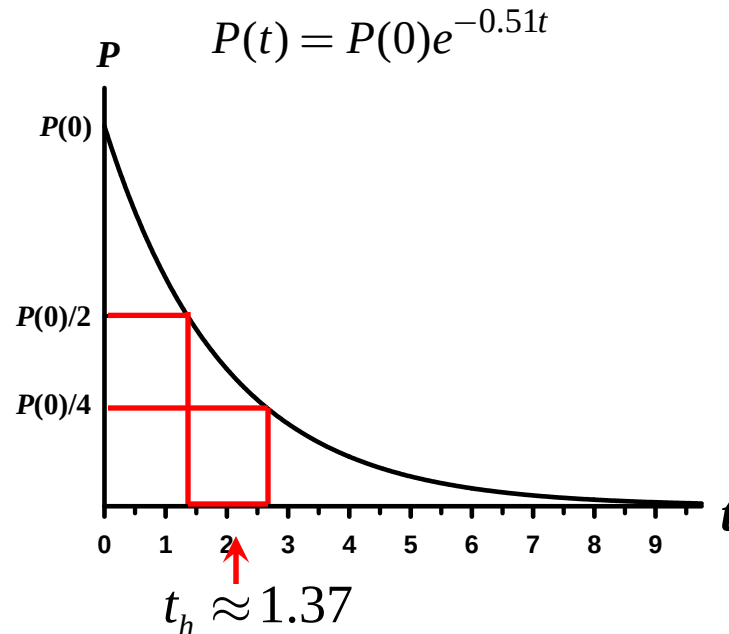
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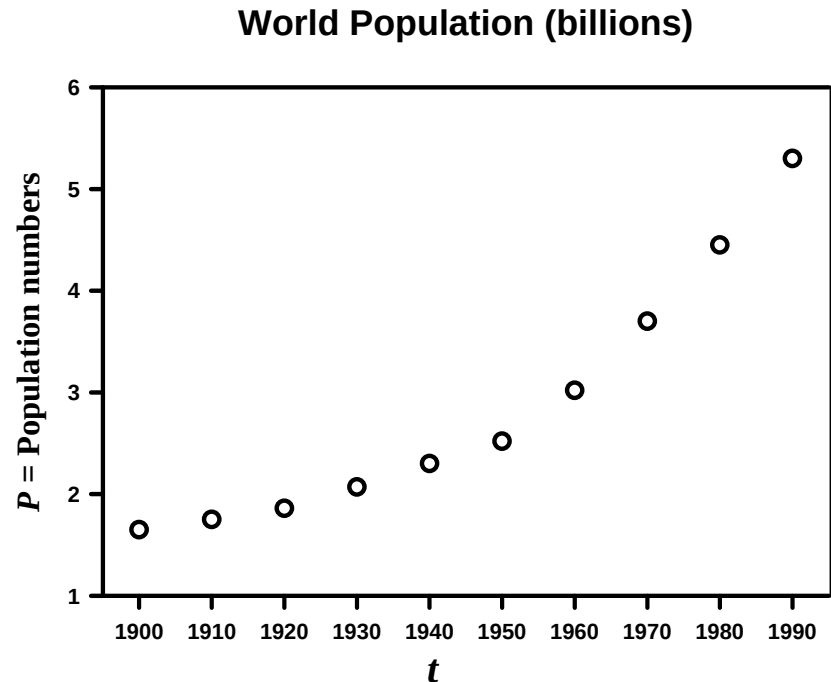


APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

EXAMPLE

Year	World Population (billions)
1900	1.655
1910	1.750
1920	1.860
1930	2.070
1940	2.300
1950	2.519
1960	3.024
1970	3.697
1980	4.442
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APPLICATIONS OF EXPONENTIAL FUNCTIONS

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Assume exponential: $P(t) = P_0 e^{rt}$

Problem: find P_0 and r

Interpolation

Force exponential to pass through
two data points

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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1900	1.655
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Assume exponential: $P(t) = P_0 e^{rt}$

Problem: find P_0 and r

Interpolation

For example, interpolate the first two data points as follows:

$$\begin{array}{lcl} P(1900) = 1.655 & & 1.655 = P_0 e^{1900r} \\ P(1910) = 1.750 & \Rightarrow & 1.750 = P_0 e^{1910r} \end{array}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

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Assume exponential: $P(t) = P_0 e^{rt}$

Problem: find P_0 and r

Interpolation

For example, interpolate the first two data points as follows:

$$1.655 = P_0 e^{1900r} \quad 1.750 = P_0 e^{1910r}$$

2 equations in 2 unknowns

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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Interpolation

For example, interpolate the first two data points as follows:

$$\begin{array}{ccc} 1.655 = P_0 e^{1900r} & & 1.750 = P_0 e^{1910r} \\ \swarrow & & \searrow \\ P_0 = 1.655 e^{-1900r} & \longrightarrow & \end{array}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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For example, interpolate the first two data points as follows:

$$\begin{array}{lcl} 1.655 = P_0 e^{1900r} & & 1.750 = P_0 e^{1910r} \\ P_0 = 1.655 e^{-1900r} & \nearrow & \\ & & 1.750 = 1.655 e^{10r} \end{array}$$

1 equation in 1 unknown

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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Assume exponential: $P(t) = P_0 e^{rt}$

Problem: find P_0 and r

Interpolation

For example, interpolate the first two data points as follows:

$$1.655 = P_0 e^{1900r} \qquad 1.750 = P_0 e^{1910r}$$


$$1.750 = 1.655e^{10r}$$

$$r = \frac{1}{10} \ln \left(\frac{1.750}{1.655} \right) \approx 5.581 \times 10^{-3}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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Assume exponential: $P(t) = P_0 e^{rt}$

Problem: find P_0 and r

Interpolation

For example, interpolate the first two data points as follows:

$$1.655 = P_0 e^{1900r} \quad 1.750 = P_0 e^{1910r}$$

$$P_0 = 1.655 e^{-1900r} \approx 4.104 \times 10^{-5}$$

$$r = \frac{1}{10} \ln \left(\frac{1.750}{1.655} \right) \approx 5.581 \times 10^{-3}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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Assume exponential: $P(t) = P_0 e^{rt}$

Problem: find P_0 and r

Interpolation

For example, interpolate the first two data points :

$$P_0 \approx 4.104 \times 10^{-5}$$

$$r \approx 5.581 \times 10^{-3}$$

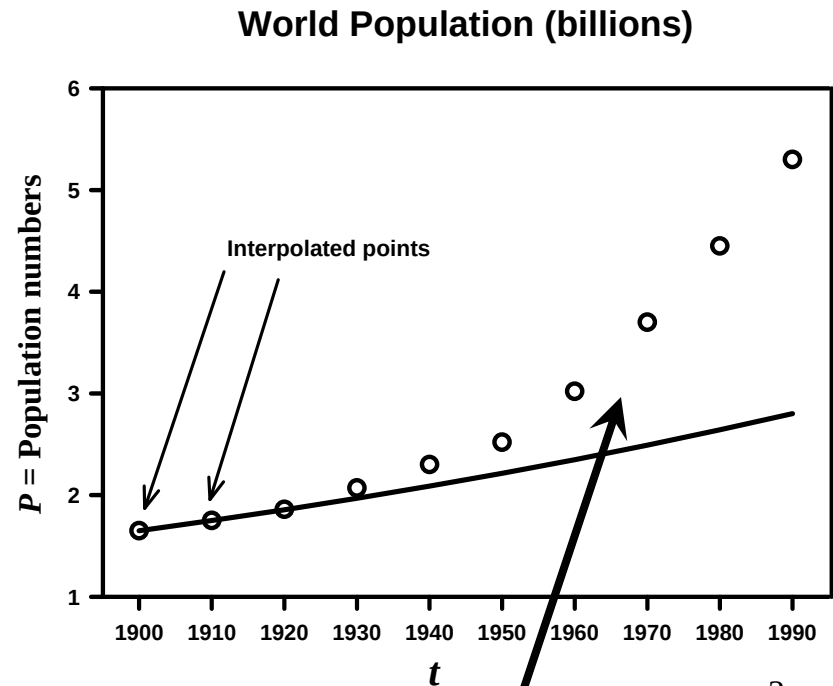
$$P(t) \approx \left(4.104 \times 10^{-5}\right) e^{(5.581 \times 10^{-3})t}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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$$P(t) \approx \left(4.104 \times 10^{-5}\right) e^{(5.581 \times 10^{-3})t}$$

Poor fit !

APPLICATIONS OF EXPONENTIAL FUNCTIONS

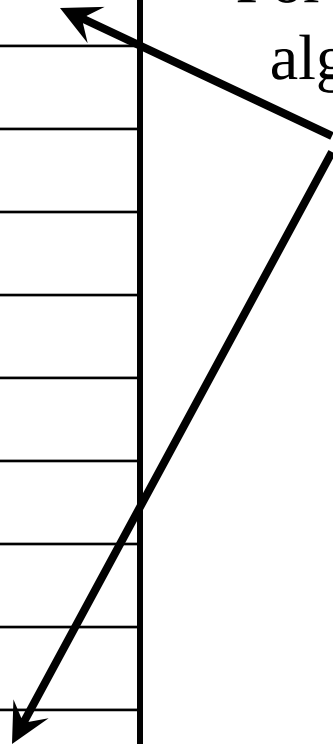
(2) Exponential Growth

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Other interpolations are possible.

For example, we could use the same algebraic procedure to interpolate the first & last data points.



APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

EXAMPLE

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Other interpolations are possible.

For example, we could use the same algebraic procedure to interpolate the first & last data points.

Turns out:

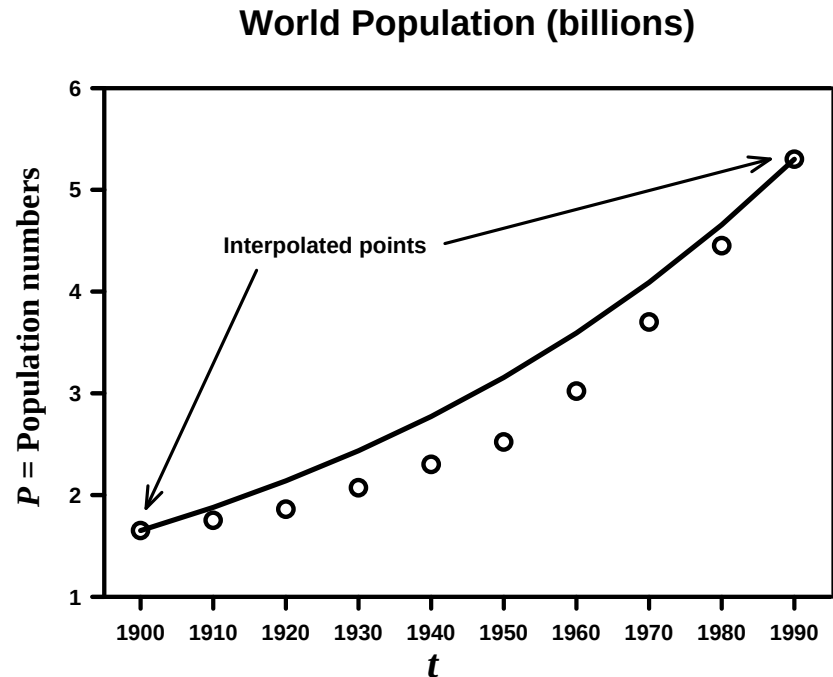
$$P(t) \approx \left(3.821 \times 10^{-11}\right) e^{(1.289 \times 10^{-2})t}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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$$P(t) \approx \left(3.821 \times 10^{-11}\right) e^{(1.289 \times 10^{-2})t}$$

Still not a good fit.

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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What is the “best” exponential fit ?

Can we use the
Method of Least Squares?

Recall ...

$$\begin{aligned} \text{error} &= (\text{data} - \text{prediction})^2 \\ &= (P_{\text{data}} - P(t_{\text{data}}))^2 \\ &= (P_{\text{data}} - P_0 e^{rt_{\text{data}}})^2 \end{aligned}$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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What is the “best” exponential fit ?

Can we use the
Method of Least Squares?

Recall ...

method minimizes the
sum total of all errors

$$E = \sum \left(P_{data} - P_0 e^{rt_{data}} \right)^2$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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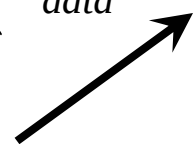
What is the “best” exponential fit ?

Can we use the
Method of Least Squares?

Recall ...

method minimizes the
sum total of all errors

$$E = \sum \left(P_{data} - P_0 e^{rt_{data}} \right)^2$$

nonlinear in P_0 and r

Nonlinear Least Squares

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

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Year	World Population (billions)
1900	1.655
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Nonlinear Least Squares Method

- Difficult to apply
- Numerical difficulties often arise

Instead, a common procedure is to search for a *linear relationship* between *transformed* variables and then use Linear Least Squares.

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

EXAMPLE

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$$P = P_0 e^{rt}$$

$$\ln P = \ln(P_0 e^{rt})$$

$$\ln P = rt + \ln P_0$$

This is a *linear* relationship between

$y = \ln P$ and t :

$$y = rt + \ln P_0$$

$r = \text{slope}$ $\ln P_0 = \text{intercept}$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

EXAMPLE

Year	World Population (ln scale)
1900	0.504
1910	0.560
1920	0.621
1930	0.728
1940	0.833
1950	0.924
1960	1.107
1970	1.308
1980	1.491
1990	1.664

We can “linearly regress” on
 $y = \ln P$ and t :

$$y = rt + \ln P_0$$

Turns out (use computer or calculator)

$$\ln P_0 \approx -24.604$$

$$r \approx 1.315 \times 10^{-2}$$

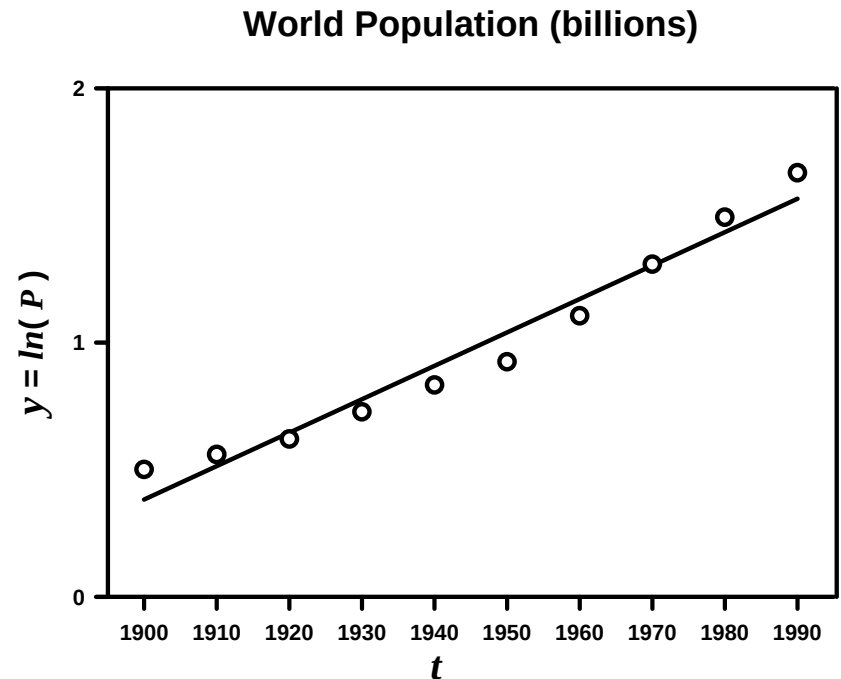
$$y = (1.315 \times 10^{-2})t - 24.604$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

EXAMPLE

Year	World Population (ln scale)
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$$y = \left(1.315 \times 10^{-2}\right)t - 24.604$$

APPLICATIONS OF EXPONENTIAL FUNCTIONS

(2) Exponential Growth

EXAMPLE

Year	World Population (billions)
1900	1.655
1910	1.750
1920	1.860
1930	2.070
1940	2.300
1950	2.519
1960	3.024
1970	3.697
1980	4.442
1990	5.280

Linear regression on a log scale ...

$$\ln P_0 \approx -24.604$$

$$r \approx 1.315 \times 10^{-2}$$

$$y = (1.315 \times 10^{-2})t - 24.604$$

In terms of the original variables ...

$$P_0 \approx 2.063 \times 10^{-11}$$

$$r \approx 1.315 \times 10^{-2}$$

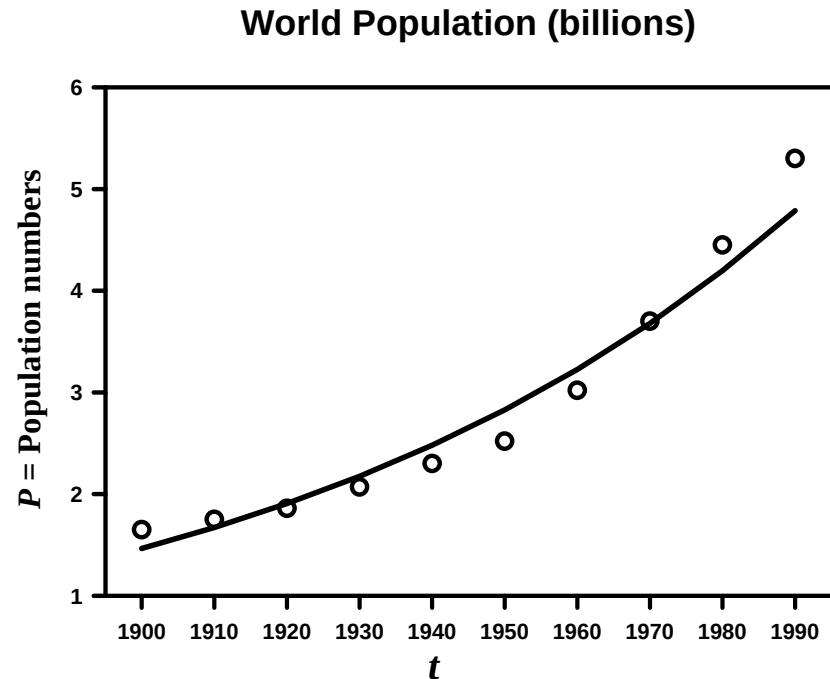
$$P(t) = P_0 e^{rt} \approx (3.30 \times 10^{-11}) e^{(1.30 \times 10^{-2})t}$$

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