

MATH 250a

Fall Semester 2007

Section 2 (J. M. Cushing)

Thursday, August 23

<http://math.arizona.edu/~cushing/250a.html>

APPLICATIONS OF LINEAR FUNCTIONS

In some applications, linear relationships are “assumed” or “hypothesized” to hold between variables.

Problems :

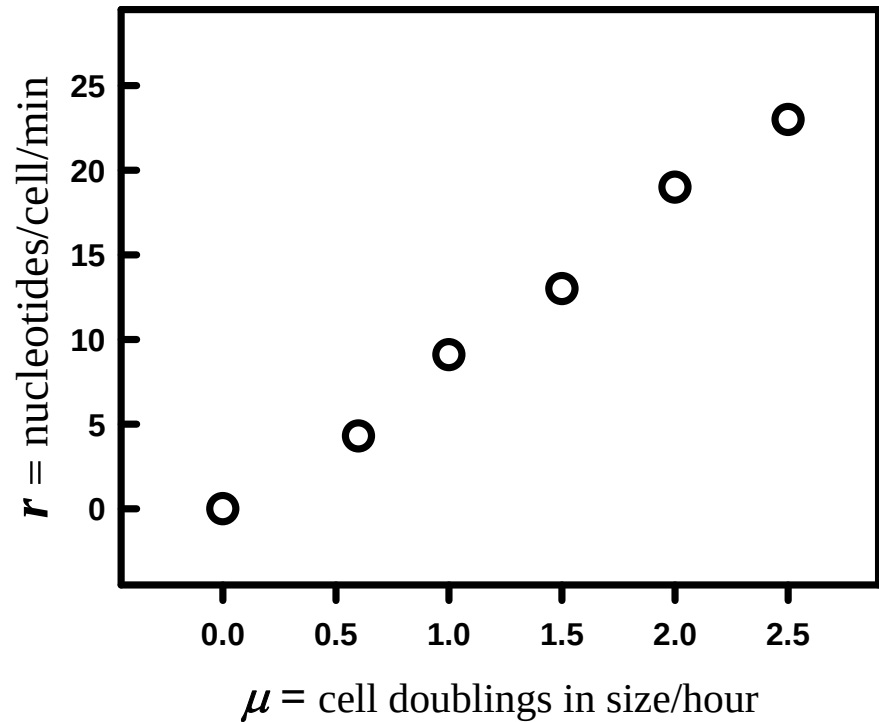
Given data, under the assumption of a linear relationship

- calculate (estimate) m and b
- accept or reject the hypothesized linear relationship

EXAMPLE 1

An experimental observation:
rate of *mRNA* production (r) depends on
length of time it takes a cell to divide (μ)

μ	r
0.0	0.0
0.6	4.3
1.0	9.1
1.5	13
2.0	19
2.5	23

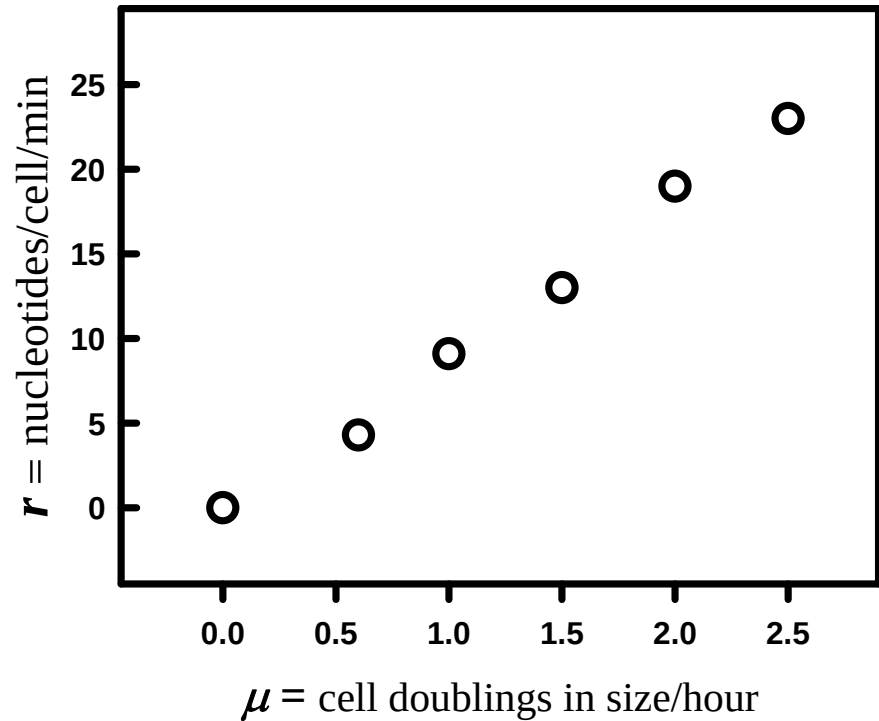


EXAMPLE 1

Relationship appears linear

Let's *hypothesize* that $r = m\mu$

μ	r
0.0	0.0
0.6	4.3
1.0	9.1
1.5	13
2.0	19
2.5	23



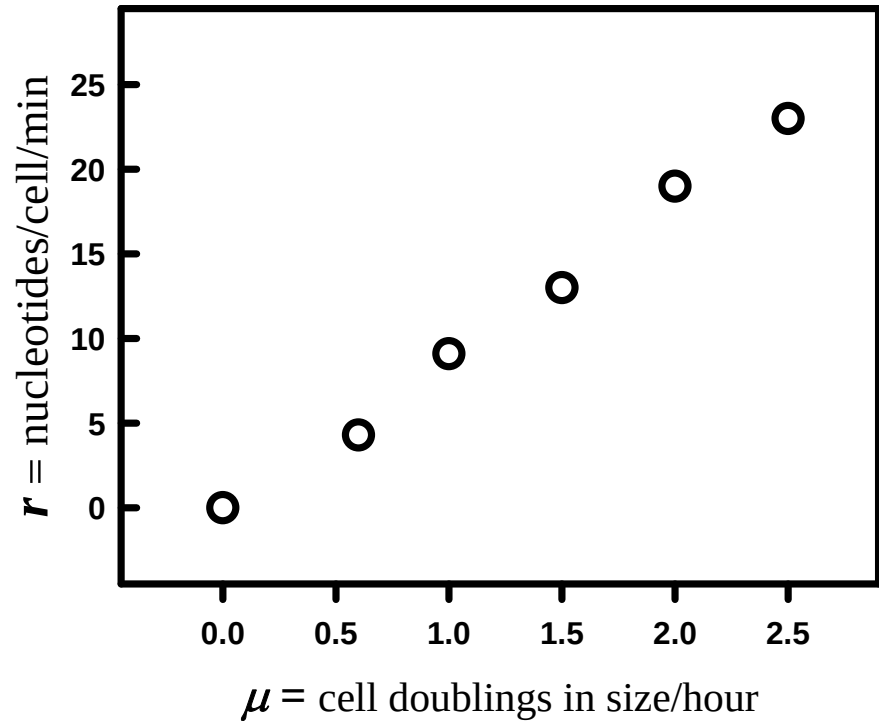
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0.6	4.3	
1.0	9.1	
1.5	13	
2.0	19	
2.5	23	

Estimate slope from the data

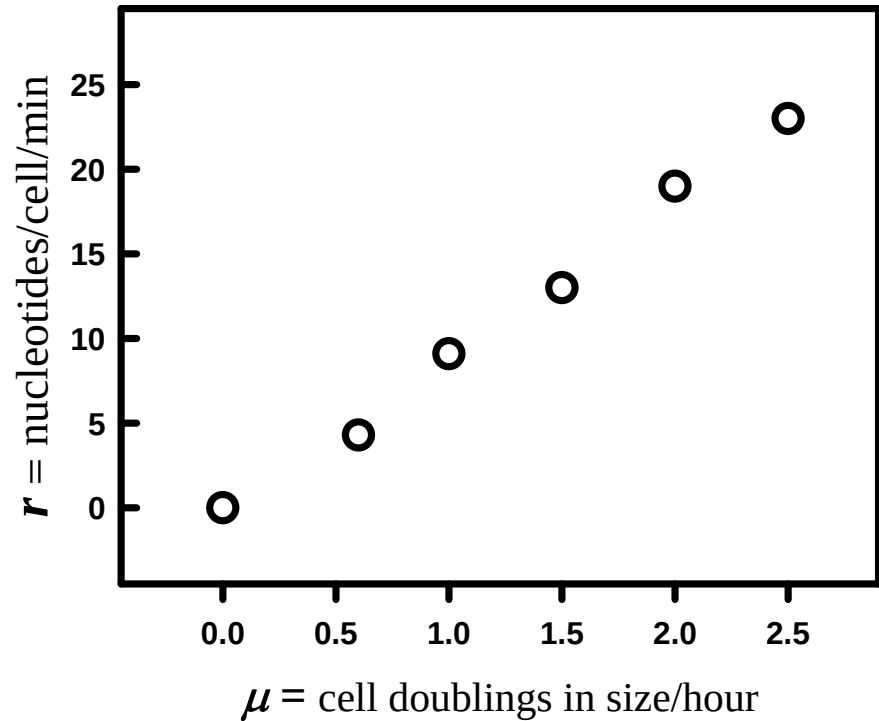


EXAMPLE 1

Relationship appears linear

Let's hypothesize that $r = m\mu$

μ	r	m
0.0	0.0	
0.6	4.3	7.17
1.0	9.1	
1.5	13	
2.0	19	
2.5	23	

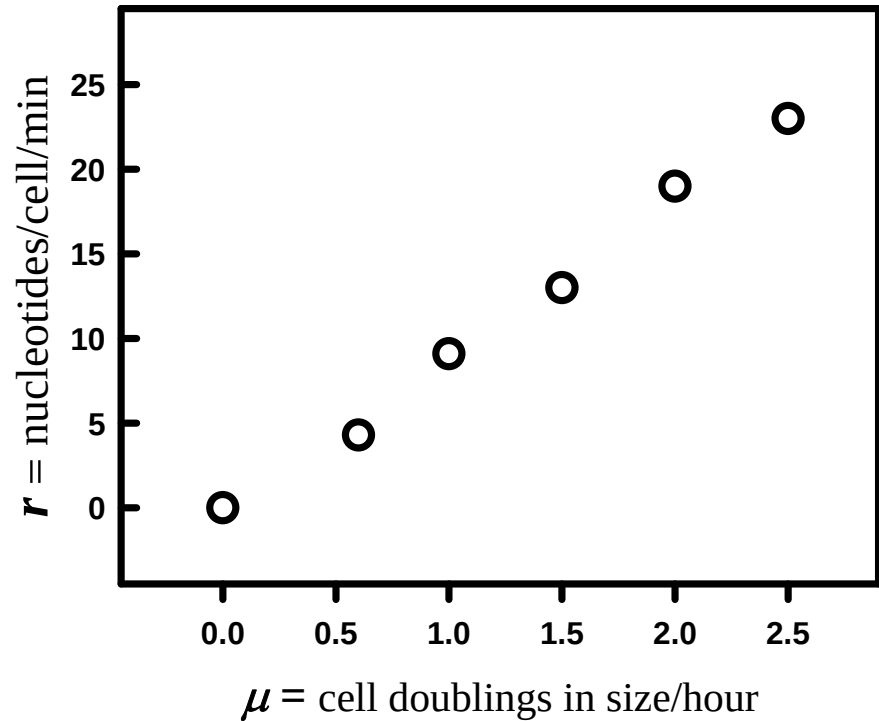


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1.0	9.1	9.10
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Let's hypothesize that $r = m\mu$

μ	r	m
0.0	0.0	
0.6	4.3	7.17
1.0	9.1	9.10
1.5	13	8.67
2.0	19	9.50
2.5	23	9.20

m is *not* the same for all data pairs !

Linearity hypothesis is clearly wrong

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2.5	23	9.20

m is *not* the same for all data pairs !

Linearity hypothesis is clearly wrong

But wait ...

perhaps the hypothesis is correct
and there are errors in the data !

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0.6	4.3	7.17
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2.5	23	9.20

If we keep the hypothesis and assume data errors, how do we

- best estimate m ?
- use result to accept or reject the hypothesis ?

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2.0	19	9.50
2.5	23	9.20

If we keep the hypothesis and assume data errors, how do we

- best estimate m ?
- use result to accept or reject the hypothesis ?

We'll consider only the first question.

Second question is studied in a statistics class.

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0.6	4.3
1.0	9.1
1.5	13
2.0	19
2.5	23

If we keep the hypothesis and assume data errors, how do we

- best estimate m ?

Approach: pick m so the
total amount of error is minimal

EXAMPLE 1

Relationship appears linear

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μ	r
0.0	0.0
0.6	4.3
1.0	9.1
1.5	13
2.0	19
2.5	23

Linear hypothesis

“predicts” r from μ

Define error = | data – prediction |

EXAMPLE 1

Relationship appears linear

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μ	r	e
0.0	0.0	
0.6	4.3	
1.0	9.1	
1.5	13	
2.0	19	
2.5	23	

Linear hypothesis

“predicts” r from μ

Define error = | data – prediction |

$$e = |r_{data} - m\mu_{data}|$$

EXAMPLE 1

Relationship appears linear

Let's hypothesize that $r = m\mu$

μ	r	e
0.0	0.0	0.0
0.6	4.3	$ 4.3 - 0.6m $
1.0	9.1	
1.5	13	
2.0	19	
2.5	23	

Linear hypothesis

“predicts” r from μ

Define error = | data – prediction |

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EXAMPLE 1

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Linear hypothesis

“predicts” r from μ

Define error = | data – prediction |

$$e = |r_{data} - m\mu_{data}|$$

EXAMPLE 1

Relationship appears linear

Let's hypothesize that $r = m\mu$

E = Total error

μ	r	e
0.0	0.0	0.0
0.6	4.3	$ 4.3 - 0.6m $
1.0	9.1	$ 9.1 - 1.0m $
1.5	13	$ 13 - 1.5m $
2.0	19	$ 19 - 2.0m $
2.5	23	$ 23 - 2.5m $

$$E(m) = \sum |r_{data} - m\mu_{data}|$$

EXAMPLE 1

Relationship appears linear

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0.0	0.0	0.0
0.6	4.3	$ 4.3 - 0.6m $
1.0	9.1	$ 9.1 - 1.0m $
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2.0	19	$ 19 - 2.0m $
2.5	23	$ 23 - 2.5m $

E = Total error

$$E(m) = \sum |r_{data} - m\mu_{data}|$$

$$E(m) = |4.3 - 0.6m| + |9.1 - m| \\ + |13 - 1.5m| + |19 - 2m| \\ + |23 - 2.5m|$$

EXAMPLE 1

Relationship appears linear

Let's hypothesize that $r = m\mu$

μ	r	e
0.0	0.0	0.0
0.6	4.3	$ 4.3 - 0.6m $
1.0	9.1	$ 9.1 - 1.0m $
1.5	13	$ 13 - 1.5m $
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E = Total error

$$E(m) = \sum |r_{data} - m\mu_{data}|$$

$$E(m) = |4.3 - 0.6m| + |9.1 - m| \\ + |13 - 1.5m| + |19 - 2m| \\ + |23 - 2.5m|$$

Calculus problem:

find the minimum of $E(m)$

REVIEW

To find the minimum of $E(m)$ for an interval of m values

(1) Find critical points : m values where $\frac{dE(m)}{dm} = 0$

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To find the minimum of $E(m)$ for an interval of m values

(1) Find critical points : m values where $\frac{dE(m)}{dm} = 0$

(2) Find local minima : e.g., critical values where $\frac{d^2E(m)}{dm^2} > 0$

REVIEW

To find the minimum of $E(m)$ for an interval of m values

- (1) Find critical points : m values where $\frac{dE(m)}{dm} = 0$
- (2) Find local minima : e.g., critical values where $\frac{d^2E(m)}{dm^2} > 0$
- (3) To determine global minimum, test endpoints against local minima

EXAMPLE 1

$$E(m) = |4.3 - 0.6m| + |9.1 - m| + |13 - 1.5m| \\ + |19 - 2m| + |23 - 2.5m|$$

for $-\infty < m < +\infty$

A technical problem: $E(m)$ is not everywhere differentiable

Why ?

EXAMPLE 1

$$E(m) = |4.3 - 0.6m| + |9.1 - m| + |13 - 1.5m| \\ + |19 - 2m| + |23 - 2.5m|$$

for $-\infty < m < +\infty$

A technical problem: $E(m)$ is not everywhere differentiable

Why ?

Because the absolute value function

$$f(x) = |x|$$

is not differentiable everywhere

(derivative does not exist at $x = 0$)

EXAMPLE 1

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For this (and other statistical reasons) we re-define error :

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for $-\infty < m < +\infty$

A technical problem: $E(m)$ is not everywhere differentiable

For this (and other statistical reasons) we re-define error :

$$error = (data - prediction)^2$$

$$= (r_{data} - m\mu_{data})^2$$



squared “residuals”

EXAMPLE 1

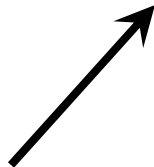
$$E(m) = |4.3 - 0.6m| + |9.1 - m| + |13 - 1.5m| \\ + |19 - 2m| + |23 - 2.5m|$$

for $-\infty < m < +\infty$

A technical problem: $E(m)$ is not everywhere differentiable

For this (and other statistical reasons) we re-define error :

$$\begin{aligned} \text{error} &= (\text{data} - \text{prediction})^2 \\ &= (r_{\text{data}} - m\mu_{\text{data}})^2 \end{aligned}$$



Differentiable in m (for all values of m)

EXAMPLE 1

Relationship appears linear

Let's hypothesize that $r = m\mu$

μ	r	e
0.0	0.0	0.0
0.6	4.3	$(4.3 - 0.6m)^2$
1.0	9.1	$(9.1 - 1.0m)^2$
1.5	13	$(13 - 1.5m)^2$
2.0	19	$(19 - 2.0m)^2$
2.5	23	$(23 - 2.5m)^2$

E = Total error

$$E(m) = \sum (r_{data} - m\mu_{data})^2$$

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0.6	4.3	$(4.3 - 0.6m)^2$
1.0	9.1	$(9.1 - 1.0m)^2$
1.5	13	$(13 - 1.5m)^2$
2.0	19	$(19 - 2.0m)^2$
2.5	23	$(23 - 2.5m)^2$

E = Total error

$$E(m) = \sum (r_{data} - m\mu_{data})^2$$

Calculus problem:

find the value of m that
minimizes $E(m)$.

Called the
Method of Least Squares

EXAMPLE 1

$$E(m) = (4.3 - 0.6m)^2 + (9.1 - m)^2 + (13 - 1.5m)^2 \\ + (19 - 2m)^2 + (23 - 2.5m)^2$$

minimize for $-\infty < m < +\infty$

EXAMPLE 1

$$E(m) = 13.86m^2 - 253.36m + 1160.3$$

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$$m_c \approx 9.14$$

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Extremal test : $\frac{d^2E(m)}{dm^2} = 27.72 > 0 \Rightarrow m_c$ is a local minimum

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Test endpoints : $\lim_{t \rightarrow +\infty} E(m)$
 $\lim_{t \rightarrow -\infty} E(m)$

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$$\lim_{t \rightarrow +\infty} E(m) = +\infty$$

Test endpoints : $\lim_{t \rightarrow -\infty} E(m)$

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Test endpoints : $\lim_{t \rightarrow -\infty} E(m) = +\infty$

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$$m_c \approx 9.14$$

Extremal test : $\frac{d^2E(m)}{dm^2} = 27.72 > 0 \Rightarrow m_c$ is a local minimum

Test endpoints :
$$\left. \begin{array}{l} \lim_{t \rightarrow +\infty} E(m) = +\infty \\ \lim_{t \rightarrow -\infty} E(m) = +\infty \\ E(m_c) \approx 2.45 \end{array} \right\} \Rightarrow m_c \text{ is a } \textit{global} \text{ minimum}$$

EXAMPLE 1

$$E(m) = 13.86m^2 - 253.36m + 1160.3$$

Critical points : $\frac{dE(m)}{dm} = 27.72m - 253.36 = 0$

$$m_c \approx 9.14$$

NOTE:

In general, global minimization test is unnecessary for the
Method of Least Squares ... why?

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$$E(m) = 13.86m^2 - 253.36m + 1160.3$$

Critical points : $\frac{dE(m)}{dm} = 27.72m - 253.36 = 0$

$$m_c \approx 9.14$$

NOTE:

In general, global minimization test is unnecessary for the
Method of Least Squares ... why?

$E(m)$ is a *quadratic* whose graph is an *upward cupped parabola*.

Therefore, there is one critical point and it is a global minimum.

EXAMPLE 1

Least Squares Best Fit

$$r = m_c \mu$$

$$r \approx 9.14\mu$$

μ	r
0.0	0.0
0.6	4.3
1.0	9.1
1.5	13
2.0	19
2.5	23

We can use this estimated linear relationship
to make predictions.

For example,

when $\mu = 0.5$, we predict $r = 4.57$

when $m = 3$, we predict $r = 27.42$

EXAMPLE 1

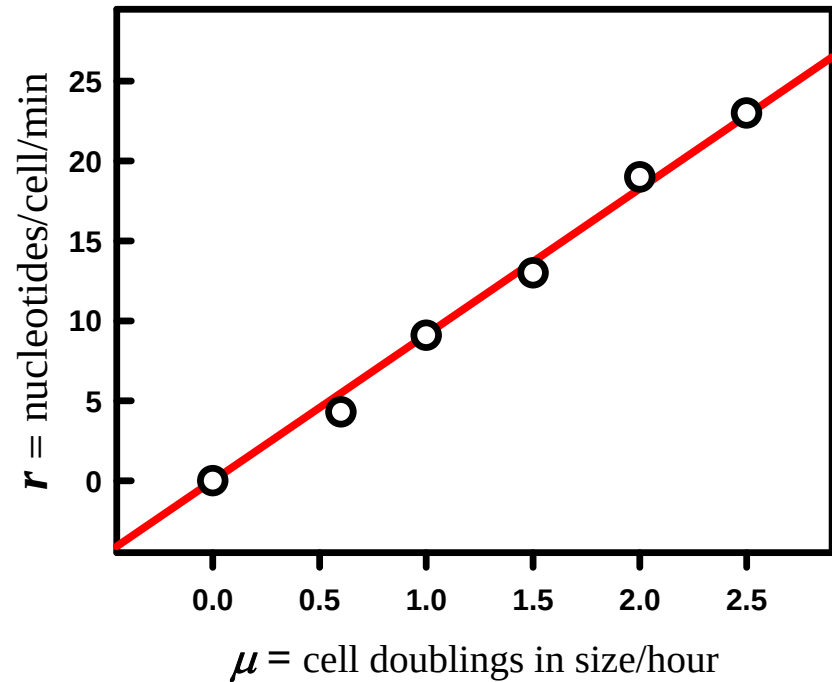
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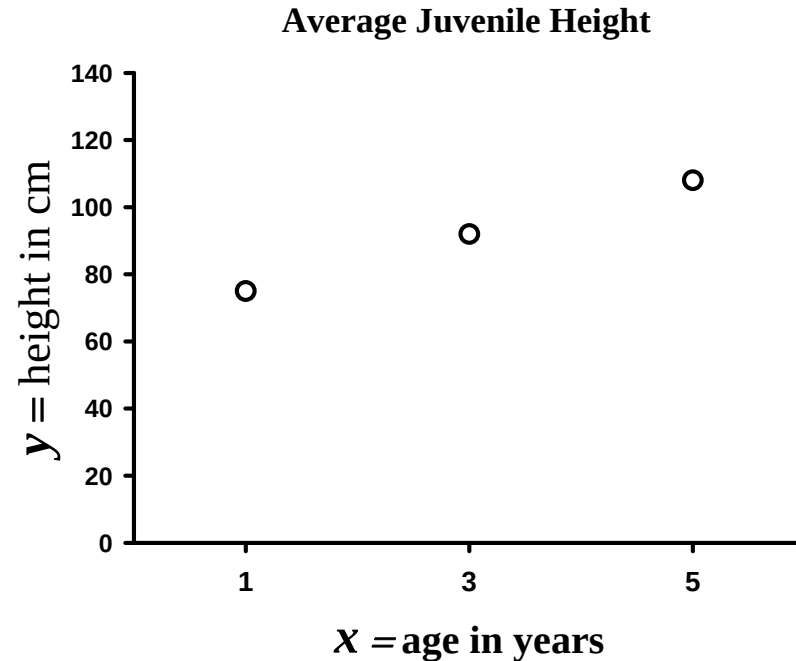
μ	r
0.0	0.0
0.6	4.3
1.0	9.1
1.5	13
2.0	19
2.5	23

LINEAR REGRESSION



EXAMPLE 2

Age x (yrs)	Juvenile Hgt y (cm)
1	75
3	92
5	108



Appear linearly related. Hypothesize that data is.

Problem: find the equation $y = mx + b$ of the “best” linear fit.

Use Method of Least Squares.

EXAMPLE 2

Age x (yrs)	Juvenile Hgt y (cm)	Prediction $mx + b$
1	75	$m + b$
3	92	$m3 + b$
5	108	$m5 + b$

EXAMPLE 2

Age x (yrs)	Juvenile Hgt y (cm)	Prediction $mx + b$	Squared residual
1	75	$m + b$	$(m+b - 75)^2$
3	92	$m3 + b$	$(m3+b - 92)^2$
5	108	$m5 + b$	$(m5+b - 108)^2$

Total Error

$$T(m,b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

Necessary condition for least square estimates of m and b
is to choose m and b so that

$$\frac{dT}{dm} = 0 \quad \text{and} \quad \frac{dT}{db} = 0$$

It turns out that these conditions are also sufficient.

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 2(m + b - 75)$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 2(m + b - 75) + 2(3m + b - 92)$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 2(m + b - 75) + 2(3m + b - 92) \times 3$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 2(m + b - 75) + 2(3m + b - 92) \times 3 + 2(5m + b - 108) \times 5$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 2(m + b - 75) + 2(3m + b - 92) \times 3 + 2(5m + b - 108) \times 5$$

$$\frac{dT}{db} = 2(m + b - 75) + 2(3m + b - 92) + 2(5m + b - 108)$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 35m + 9b - 891$$

$$\frac{dT}{db} = 9m + 3b - 275$$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

$$\frac{dT}{dm} = 35m + 9b - 891$$

$$\frac{dT}{db} = 9m + 3b - 275$$

Requirements	$\frac{dT}{dm} = 0$	become	$35m + 9b = 891$
	$\frac{dT}{db} = 0$		$9m + 3b = 275$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m,b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

Two algebraic equations in
two unknowns
$$\begin{cases} 35m + 9b = 891 \\ 9m + 3b = 275 \end{cases}$$

Solve for $m = 8.25, \quad b = 66.92$

EXAMPLE 2

Method of Least Squares:

Choose m and b so as to minimize

$$T(m, b) = (m + b - 75)^2 + (3m + b - 92)^2 + (5m + b - 108)^2$$

Two algebraic equations in
two unknowns
$$\begin{cases} 35m + 9b = 891 \\ 9m + 3b = 275 \end{cases}$$

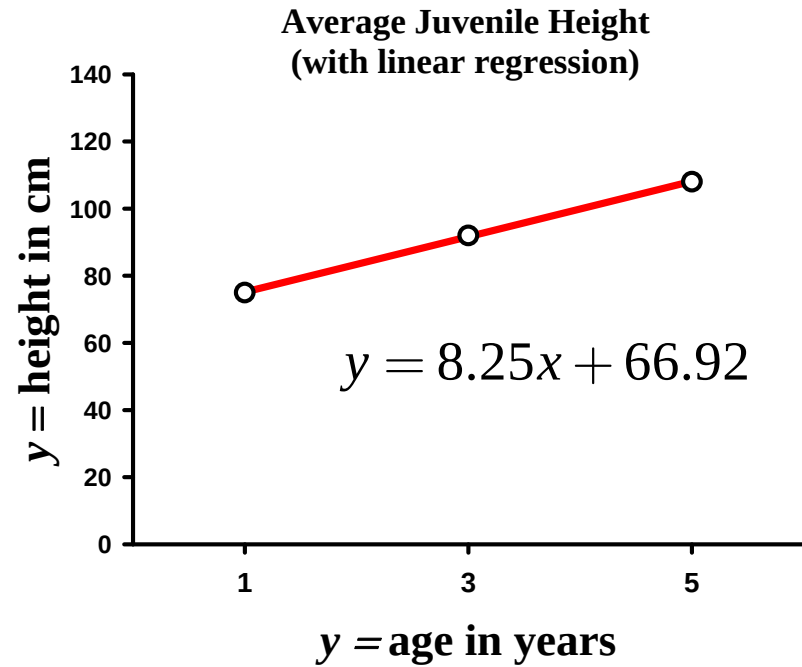
Solve for $m = 8.25$, $b = 66.92$

Linear regression line:

$$y = 8.25x + 66.92$$

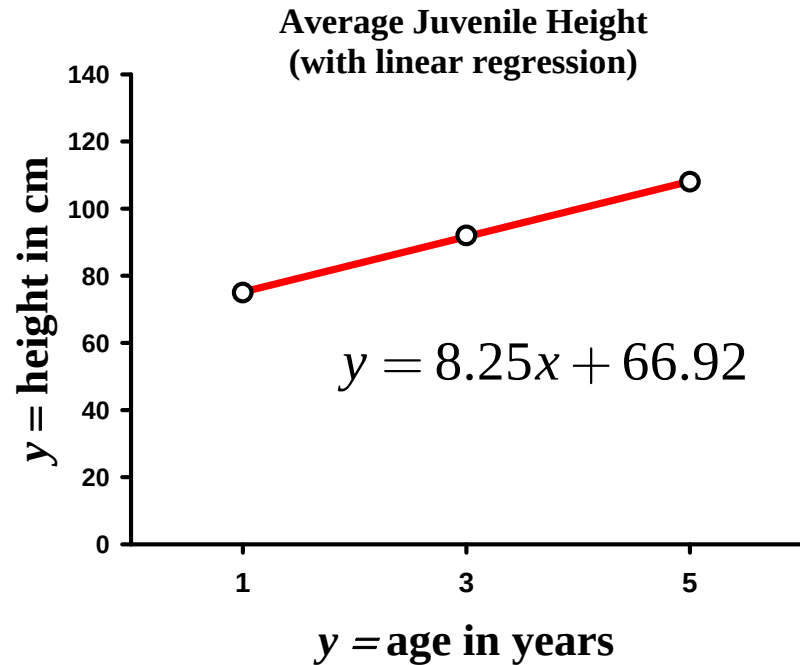
EXAMPLE 2

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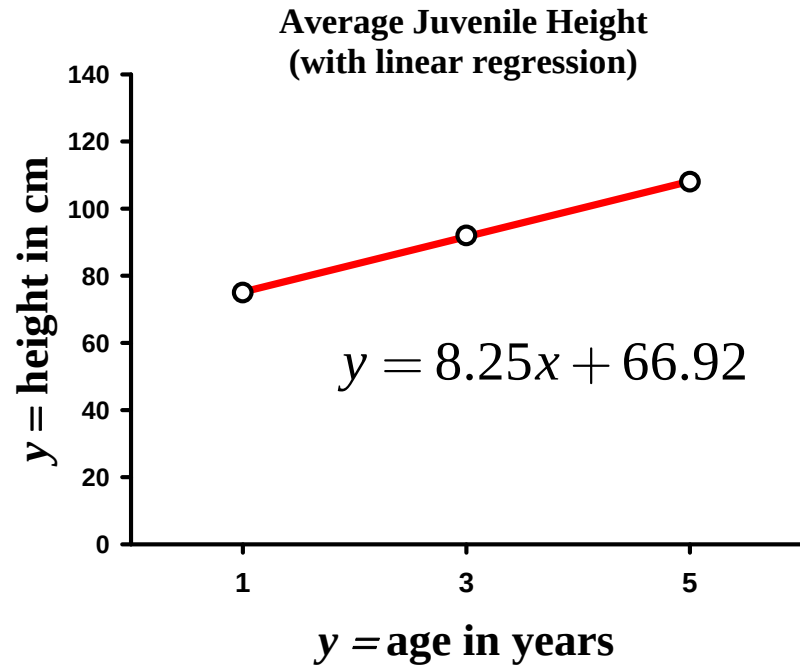


Prediction at age $x = 7$ is

$$y = 8.25 \times 7 + 66.92 = 124.88$$

EXAMPLE 2

Age x (yrs)	Juvenile Hgt y (cm)
1	75
3	92
5	108
7	121

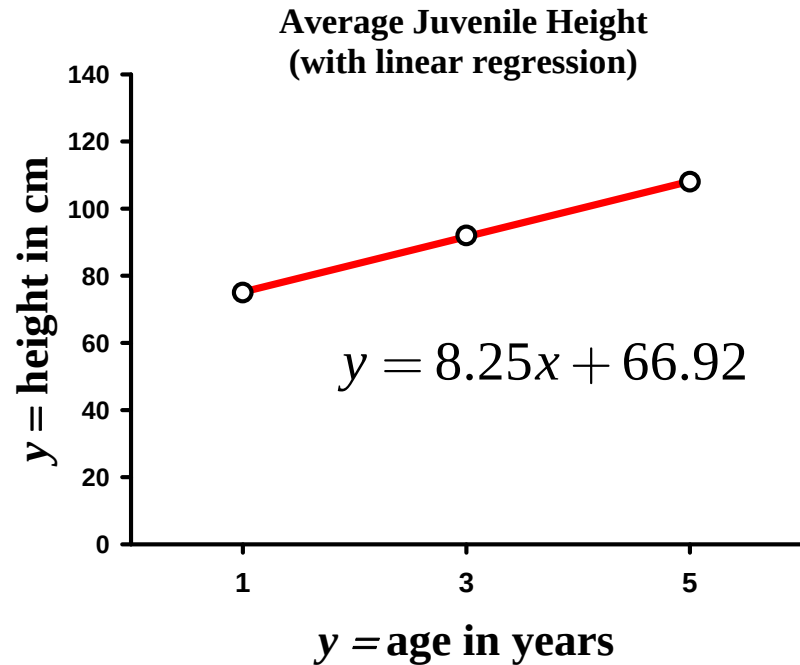


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EXAMPLE 2

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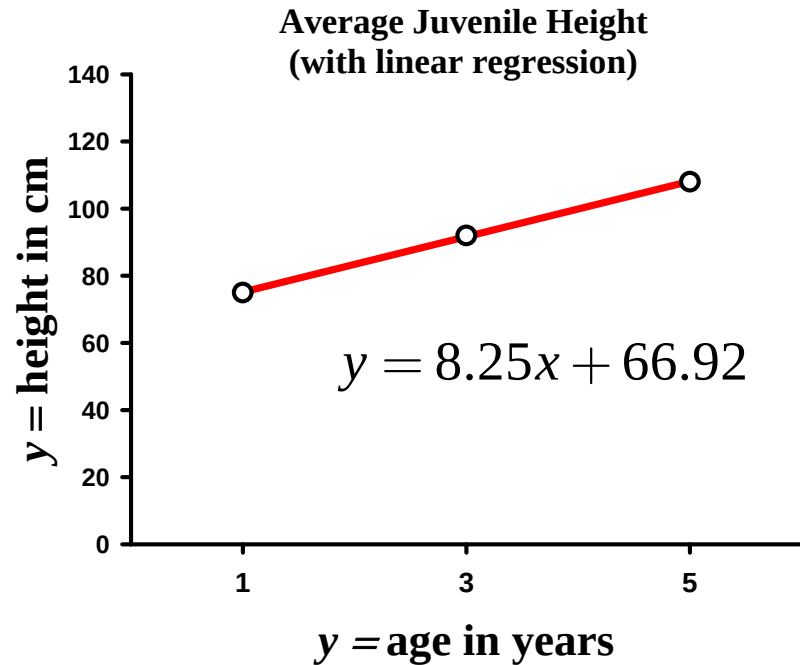


Prediction at age $x = 9$ is

$$y = 8.25 \times 9 + 66.92 = 141.17$$

EXAMPLE 2

Age x (yrs)	Juvenile Hgt y (cm)
1	75
3	92
5	108
7	121
9	130

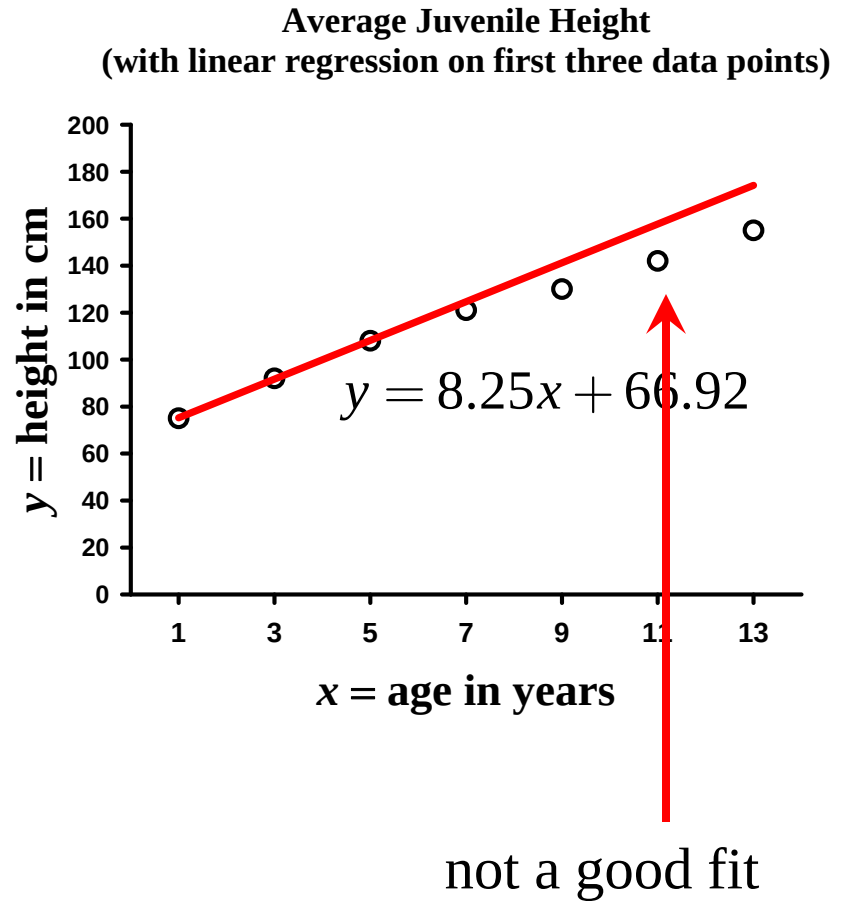


Prediction at age $x = 9$ is

$$y = 8.25 \times 9 + 66.92 = 141.17$$

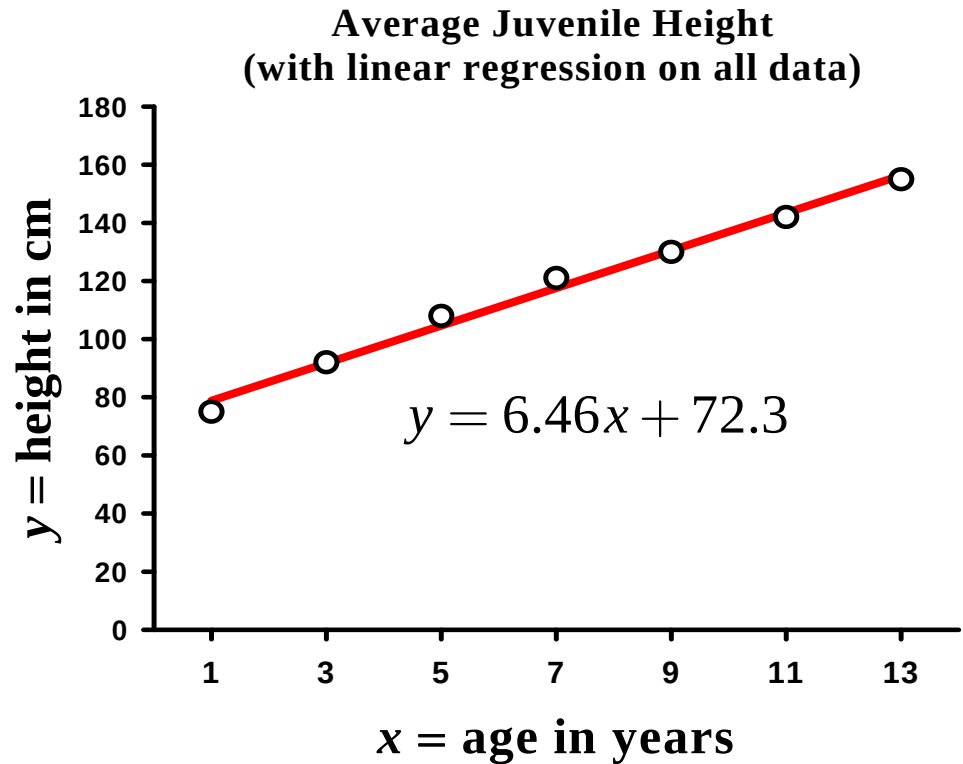
EXAMPLE 2

Age x (yrs)	Juvenile Hgt y (cm)
1	75
3	92
5	108
7	121
9	130
11	142
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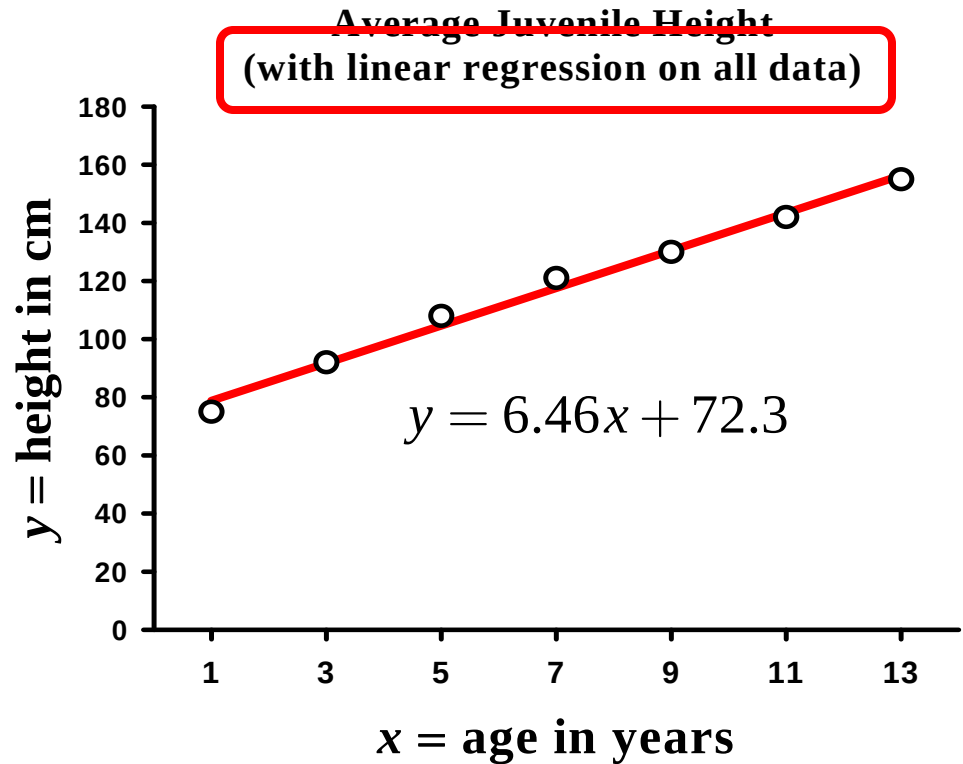
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Chapter 1.2 – 1.4

Exponential and Logarithmic Functions

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Example

Deposit \$100 in a bank account at 5% interest (compounded annually).

What is the account worth in 5 years ? 10 years ? t years ?

Chapter 1.2 – 1.4

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$P(t)$ = balance after t years

$P(0) = 100$

Chapter 1.2 – 1.4

Exponential and Logarithmic Functions

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next year's balance = $1.05 \times$ this year's balance

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Chapter 1.2 – 1.4

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(mathematical induction)

Chapter 1.2 – 1.4

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$$P(5) = 1.05^5 \times 100 \approx 127.63$$

$$P(10) = 1.05^{10} \times 100 \approx 162.89$$

Chapter 1.2 – 1.4

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An example of an *exponential function*
... with base 1.05.

Other bases occur for other interest rates.

Chapter 1.2 – 1.4

Exponential and Logarithmic Functions

Example

Deposit $P(0)$ in a bank account at $r\%$ interest (compounded annually).
What is the account worth in t years ?

Chapter 1.2 – 1.4

Exponential and Logarithmic Functions

Example

Deposit $P(0)$ in a bank account at $r\%$ interest (compounded annually).

What is the account worth in t years ?

next year's balance $= (1 + r) \times$ this year's balance

$$P(t + 1) = (1 + r)P(t)$$

$$P(t) = (1 + r)^t P(0)$$

... an “exponential function” with base $1 + r$.

DEFINITION (p. 11 text):

$P(t)$ is an *exponential function* of base $a > 0$ if

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$$a^t a^s = a^{t+s}$$

$$(a^t)^s = a^{ts}$$

$$(ab)^t = a^t b^t$$

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NOTE: $(a + b)^t \neq a^t + b^t$ (a common error)

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Solve for t :

$$(1 + r)^t = 2$$

This requires the *inverse* of the exponential function,
that is, requires the *logarithm function*