

4/30/10

Math 250B

ex Van der Pol oscillator

$$\square \text{ Vander Pol Equation: } \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$$

[nonlinear, 2nd order homogeneous ODE]

fixed point
= critical point
= equilb. solution

$$\square \text{ as a system of 1st order ODEs: } \begin{cases} \dot{x} = y \\ \dot{y} = -x + y(1-x^2) \end{cases} \quad (\text{letting } \mu=1)$$

$\square$ equilibria? $\rightarrow$ clearly $y=0$ (such that $\dot{x}=0$ for all t), so as a consequence, we must have $x=0$ too

$$(x_{eq}, y_{eq}) = (0, 0) \quad [\text{only equilibrium point}]$$

$\square$ local stability? $\rightarrow$ determine the Jacobian about the fixed point

this idea stems from using a (*) first-order Taylor polynomial to approximate 'local' behavior

$$J(x, y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \rightarrow J(0, 0) = \begin{bmatrix} 0 & 1 \\ -1 & 1-x^2 \end{bmatrix} \Big|_{\substack{x=0 \\ y=0}}$$

where $f(x, y) = y$
 $g(x, y) = -x + y(1-x^2)$

(from our ODEs!)

$$= \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \quad \begin{matrix} \text{Tr} = 1 \\ \text{det} = 1 \end{matrix}$$

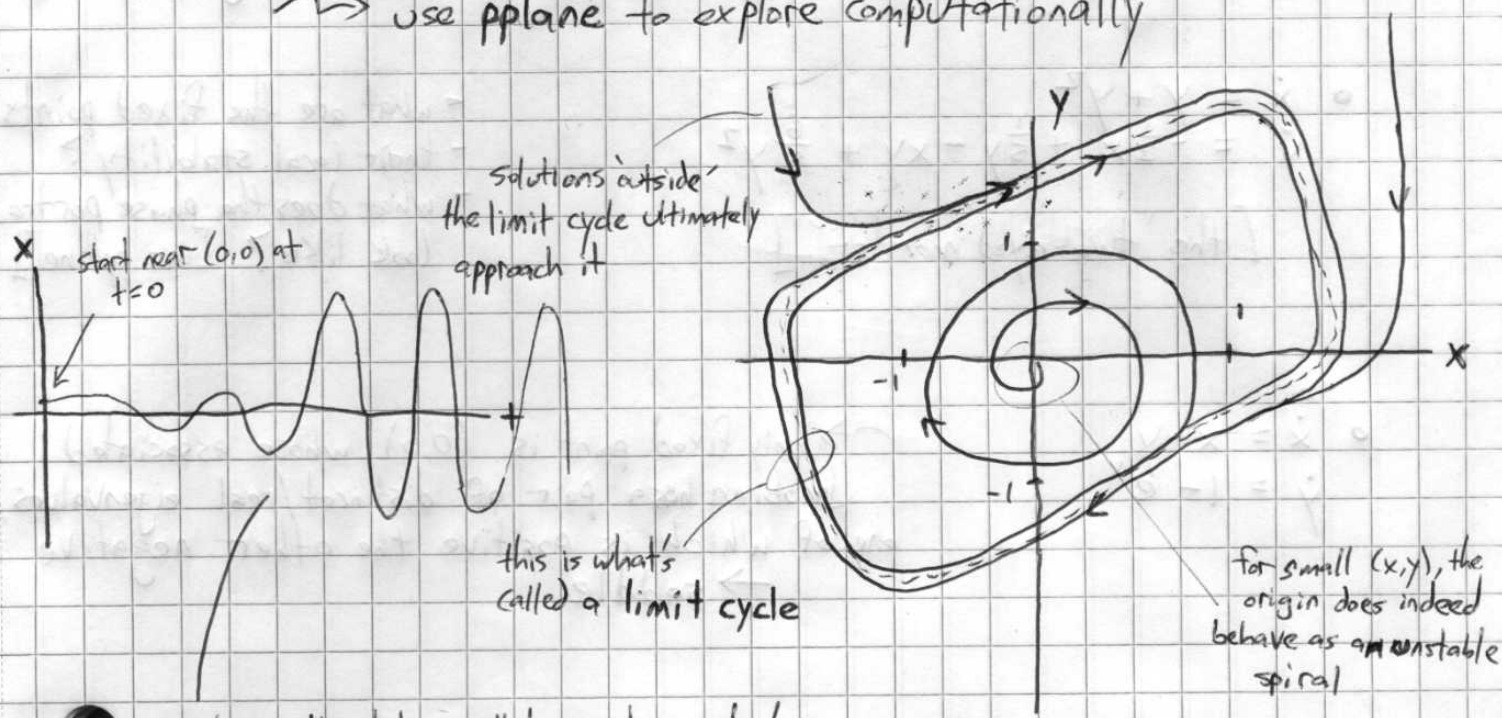
$\leadsto$ associated eigenvalues: $\lambda^2 - \text{Tr} \lambda + \text{det} = 0$

$$\lambda^2 - \lambda + 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{3}i}{2} = \frac{1}{2} \pm i\sqrt{\frac{3}{4}}$$

$\Rightarrow$ complex conjugate pair w/ positive real part: the fixed (at least 'locally') behaves as an unstable spiral

□ So what happens when you are not 'local' [i.e. x and y are not small and thus close to $(0,0)$]??

→ use pplane to explore computationally



solutions ultimately oscillate spontaneously!
(this is what happens when solutions approach a limit cycle)

- limit cycles only manifest in nonlinear systems; linear systems do not exhibit them
- long history of applications that van der Pol oscillators are used to model in both biological (e.g. auditory system) and engineering applications (e.g. signal entrainment in vacuum tubes)
- can drive w/ an external stimulus (making the eqn. non-homogeneous) such as an external tone:

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = F_0 \cos(\omega t)$$

→ if ω is chosen to be close to the 'characteristic' freq. of the oscillator (associated w/ the limit cycle), various complex phenomena can arise

A/30/10 (cont.)

□ some other nonlinear systems you can examine: [ref. Strogatz, 1994]

$$\begin{aligned} \dot{x} &= y + y^2 \\ \dot{y} &= -\frac{1}{2}x + \frac{1}{5}y - xy + \frac{6}{5}y^2 \end{aligned}$$

[the 'two-eyed monster']

- what are the fixed points?
- their local stability?
- what does the phase portrait look like? [use pplane]

$$\begin{aligned} \dot{x} &= x - y \\ \dot{y} &= 1 - e^x \end{aligned}$$

→ only fixed point is $(0,0)$, whose associated Jacobian has a pair of distinct/real eigenvalues, one of which is positive the other negative
→ saddle

$$\ddot{x} + b\dot{x} + \sin(x) = 0$$

→ rewrite as the system: $\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\sin(x) - by \end{aligned}$

[nonlinear pendulum]

⊛ [try $b=0.2$ and zoom out in pplane a bit such that $x \in [-12, 12]$, $y \in [-4, 12]$; can you connect the various solution curves to your physical intuition as to what is going on?]