

LECTURE 26

$$\text{ex } x' = 1 - x - 4 \frac{xy}{x+y}$$

$$y' = 4 \frac{xy}{x+y} - 2y$$

(HIV/AIDS) 4/28/10

NONLINEAR SYSTEMS

$$\text{ex } x'' + \alpha(x^2 - 1)x' + x = 0 \quad (\text{vander Pol})$$

- typically quite hard to solve analytically for a solution
- however, painting a qualitative picture in the phase plane can give vast insight into the behavior of the systems solutions
 - use the phase plane techniques developed for linear systems

NOTE: certain types of nonlinear DEs lend themselves well to analytical methods (i.e. separation of variables), but these are relatively rare and are really only applicable for first order eqns.

→ we are primarily interested here in systems

BASIC APPROACH

- 1 find the equilibrium points (i.e. constant solution pairs)
- 2 linearize about the fixed points → find the 'Jacobian'
- 3 analyze the linearization (as we had done for linear systems)
- 4 take a step back and try to draw general conclusions about the system's overall behavior

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LECTURE 26 (cont.)

STEP 1 find the equilibria

$$x' = f(x, y)$$

$$y' = g(x, y)$$

equilibria are
where

$$f(x, y) = 0$$

$$g(x, y) = 0$$

not always easy to do and could involve some intensive algebra

ex Predator-Prey model

$$x' = (y-1)x$$

$$y' = (1-x)y$$

$$(y-1)x = 0$$

$$(1-x)y = 0$$

(equilibrium solutions need
to satisfy both eqns.)

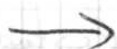
two equilibrium
solutions

$$\rightarrow \begin{cases} (x, y) = (0, 0) \\ (x, y) = (1, 1) \end{cases}$$

ex AIDS/HIV

$$x' = 1 - x - 4 \frac{xy}{x+y}$$

$$y' = 4 \frac{xy}{x+y} - 2y$$



$$1 - x - 4 \frac{xy}{x+y} = 0$$

$$4 \frac{xy}{x+y} - 2y = 0$$

clearly $(x, y) = (1, 0)$ will work, but any else?

take bottom equation to find $x = y$, plug into top eqn. to find $x = \frac{1}{3}$

$$\rightarrow (x, y) = \left(\frac{1}{3}, \frac{1}{3}\right) \text{ also an equilibrium}$$

STEP 2

Linearize so as to find the phase plane portrait near each equilibrium

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IDEA

approximate the nonlinear system near an equilibrium by a linear system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

nonlinear system

$$\Rightarrow \hat{x}' = A \hat{x}$$

linearized system
about equilibrium
point (x_e, y_e)

$$\text{where } A \equiv J(x, y) = \begin{bmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{bmatrix} \bigg|_{(x_e, y_e)}$$

Jacobian matrix
(evaluated @ (x_e, y_e))

EX

$$x' = (y-1)x$$

$$y' = (1-x)y$$

$$\} (x_e, y_e) = (0, 0), (1, 1)$$

$$J = \begin{bmatrix} \frac{df}{dx} & \frac{df}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{bmatrix} = \begin{bmatrix} y-1 & x \\ -y & 1-x \end{bmatrix}$$

$$\lambda = 1 \quad \hat{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

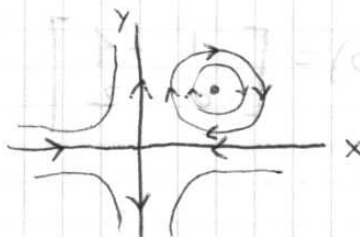
$$\lambda = -1 \quad \hat{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

STEP 3

$$J(0, 0) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{find } \lambda} \lambda^2 - 1 = 0, \lambda = \pm 1 \quad (\text{unstable saddle})$$

$$J(1, 1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightarrow \lambda^2 + 1 = 0, \lambda = \pm i \quad (\text{center, neutrally stable})$$

⊛ $\Rightarrow$ now we can paint a qualitative picture of what's going on in the phase plane (and thus the behavior of some solutions)



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LECTURE 26 (cont.)

Linearization + Stability

• similar to linear (autonomous, homogeneous) case, the real part of the eigenvalue's sign tells you the long term behavior

⇒ if both eigenvalues (λ) of the Jacobian $J(x_e, y_e)$ satisfy $\text{Re } \lambda < 0$, then the equilibrium is stable

(i.e. nearby orbits stay near and tend to equilibrium as $t \rightarrow \infty$)

⇒ if at least one eigenvalue of $J(x_e, y_e)$ satisfies $\text{Re } \lambda > 0$, then the equilibrium is unstable

HIV/AIDS

$$x' = 1 - x - 4 \frac{xy}{x+y}$$

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[see a couple pgs. back for this derivation]

$$\Rightarrow (x_e, y_e) = (1, 0), \left(\frac{1}{3}, \frac{1}{3}\right)$$

$$J(x_e, y_e) = \begin{bmatrix} \frac{df}{dx} \big|_{(x_e, y_e)} & \frac{df}{dy} \big|_{(x_e, y_e)} \\ \frac{dg}{dx} \big|_{(x_e, y_e)} & \frac{dg}{dy} \big|_{(x_e, y_e)} \end{bmatrix} = \begin{bmatrix} -1 - 4 \left(\frac{y(x+y) - xy}{(x+y)^2} \right) & -4 \left(\frac{x(x+y) - xy}{(x+y)^2} \right) \\ 4 \left(\frac{y(x+y) - xy}{(x+y)^2} \right) & -2 + 4 \left(\frac{x(x+y) - xy}{(x+y)^2} \right) \end{bmatrix}$$

$$J(1, 0) = \begin{bmatrix} -1 & -4 \\ 0 & 2 \end{bmatrix} \xrightarrow{\text{find } \lambda} \lambda^2 - \lambda - 2 = 0, \quad \lambda = -1, 2$$

$$\lambda = -1 \quad \hat{v} = \begin{bmatrix} c \\ 0 \end{bmatrix} \quad \lambda = 2 \quad \hat{w} = \begin{bmatrix} 1 \\ -3/4 \end{bmatrix}$$

Unstable saddle

$$J\left(\frac{1}{3}, \frac{1}{3}\right) = \begin{bmatrix} -2 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow \lambda^2 + 3\lambda + 3 = 0, \quad \lambda = \frac{-3 \pm i\sqrt{3}}{2}$$

$\text{Re } \lambda < 0 \Rightarrow \text{stable}$

So we know that $(1,0)$ is an unstable saddle and $(\frac{1}{3}, \frac{1}{3})$ is a stable sink

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$$\frac{(x+y)-xy}{(x+y)^2}$$

$$\frac{(x+y)-xy}{(x+y)^2}$$

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