

Examples Dealing w/ MUCGuideline (1)

To find a particular solution to the ODE

$$ay'' + by' + cy = P_m(t) e^{rt} \quad [y' \equiv \frac{dy}{dt}]$$

where $P_m(t)$ is a polynomial of order m , use the form

$$y_p(t) = t^s [A_m t^m + \dots + A_1 t + A_0] e^{rt}$$

where $s = \begin{cases} 0 & \text{if } r \text{ is not a root (i.e. eigenvalue) for the homogeneous eqn.} \\ 1 & \text{if } r \text{ is a 'simple root' (i.e. one of two distinct roots)} \\ 2 & \text{if } r \text{ is a double root} \end{cases}$

Guideline (2)

To find a particular solution to the ODE

$$ay'' + by' + cy = P_m(t) e^{\alpha t} \cos \beta t + Q_n(t) e^{\alpha t} \sin \beta t$$

where $P_m(t), Q_n(t)$ are polynomials of order m, n (respectively), then try a solution of the form

$$y_p(t) = t^s (A_k t^k + \dots + A_1 t + A_0) e^{\alpha t} \cos \beta t \\ + t^s (B_k t^k + \dots + B_1 t + B_0) e^{\alpha t} \sin \beta t$$

where k is the larger of m and n . If $\alpha + i\beta$ is an eigenvalue of y_h , then $s=1$. If $\alpha + i\beta$ is NOT an eigenvalue, let $s=0$

□ both these 'guidelines' * provide a means to ~~find~~ find a particular solution (and thus ultimately the general solution), but there are a few caveats:

- note that the arguments of the exponentials and sinusoids are linear in t
- we still need to determine the various parameters $A_k, \dots, A_1, A_0, B_k, \dots, B_1, B_0$ (i.e. the undetermined coefficients!)
- the arguments for both sinusoids (and corresponding exponentials) must be the same
- guideline #1 follows from the more general guideline #2 for $B=0$

ASIDE

* - this is just a 'guideline' and we are working under the assumption it is true (it is); you could imagine framing it a different way - our method is really just a conjecture, which we could go about trying to rigorously prove, thereby making it a theorem
[this logical flow is a bedrock of mathematics]

ex Find the form for y_p for the following ODE:

$$y'' + 2y' + 2y = 5e^{-t} \sin t + 5t^3 e^{-t} \cos t$$

→ the characteristic eqn. for the homogeneous eqn. is

$$\lambda^2 + 2\lambda + 2 = 0 \rightarrow \lambda = -1 \pm i$$

which corresponds to the form of the non-homogeneous term! so to get a linearly independent solution for y_p (re y_h), we multiply by t as the 'guideline' suggests!

to keep going, you'd plug back into the ODE and solve for the various coefficients

$$y_p(t) = t[A_3 t^3 + A_2 t^2 + A_1 t + A_0]e^{-t} \cos t \\ + t[B_3 t^3 + B_2 t^2 + B_1 t + B_0]e^{-t} \sin t$$

∴

[ex] find the general solution to

$$y'' - y = 8te^t + 2e^t$$

□ homogeneous part first: $y_h'' - y_h = 0 \rightarrow \lambda^2 - 1 = 0, \lambda = \pm 1$

$$\rightarrow y_h(t) = C_1 e^t + C_2 e^{-t}$$

□ since $\lambda = 1$ is an eigenvalue, we need an extra factor of t in y_p so to guarantee the functions comprising it are LI re those making up y_h , so we try

$$y_p(t) = t[A_1 t + A_0]e^t \quad [\text{now plug back into the ODE}]$$

$$y_p' = [A_1 t^2 + (2A_1 + A_0)t + A_0]e^t$$

$$y_p'' = [A_1 t^2 + (4A_1 + A_0)t + (2A_1 + 2A_0)]e^t$$

skipped over some
straight-forward
algebraic steps
here

$$\leadsto y_p'' - y_p = [4A_1 t + (2A_1 + 2A_0)]e^t = (8t + 2)e^t$$

$\leadsto$ two eqns., two unknowns: $A_1 = 2, A_0 = -1$

$$\text{so } y_p(t) = (2t^2 - t)e^t$$

$\Rightarrow$ thus our general solution is $y(t) = C_1 e^t + C_2 e^{-t} + (2t^2 - t)e^t$