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Math 250B

Method of Undetermined Coefficients (L.L. Ch. 7.2)

GOAL: develop a method to find the general solution to linear, 2nd order ODEs w/ const. coefficients BUT that have a 'forcing function' (i.e. are non-homogeneous)

$$a\ddot{x} + b\dot{x} + cx = f(t) \quad \text{forcing function}$$

Caveat: We will develop an approach for such called the Method of Undetermined Coefficients (MUC), however it will only work if the forcing function has one of the following properties:

- exponential function
- polynomial
- sinusoid
- products or sums of the above

□ as an aside, you already have some physical intuition for a practical example along these lines: the damped, driven harmonic oscillator (see 3/26 and 3/29 notes):

$$m\ddot{x} + b\dot{x} + kx = F(t)$$

'driving term', such as a sinusoidal 'push' (like pushing someone on a swing)

→ if $F(t)$ is sinusoidal and close to the natural freq. of the system ($\omega_0 = \sqrt{k/m}$), you can get RESONANCE

(we'll come back to this later)

GIST:

- we already know how to solve the homogeneous eqn. for x_h
- the general solution is $x_g = x_h + x_p$, where x_p is a 'particular' solution
- MUC has us essentially guessing a solution x_p that is similar in form to the forcing function, but needs to be linearly independent from the functions comprising x_h

⊗ MUC can also work for single non-homogeneous first order ODEs or systems of first order ODEs (though will stay focused on 2nd order ODEs here for clarity)

□ let's start w/ an illustrating example, consider:

$$m\ddot{x} + kx = \alpha \sin \beta t$$

[harmonic oscillator, no damping, sinusoidal driving force]

- now the general solution will be $x(t) = x_h + x_p$

- x_h is simply the solution to $m\ddot{x} + kx = 0$, which we know to be (see 3/26 notes)

$$x_h(t) = C_1 \sin \omega_0 t + C_2 \cos \omega_0 t \quad \text{where } \omega_0 = \sqrt{k/m}$$

- so now we just need to figure out $x_p(t)$. one possible form could be

$$x_p(t) = k_1 \sin \beta t + k_2 \cos \beta t \quad (?)$$

because:

- if we plug it in, we might expect something 'leftover' of the form $\sin \beta t$ (i.e. the non-homogeneous term)
- it is relatively simple

→ plug into ODE and see if it works (we don't know at present if it will!)

$$m\ddot{x} + kx = -\beta^2 m [k_1 \sin \beta t + k_2 \cos \beta t] + k [k_1 \sin \beta t + k_2 \cos \beta t] = \alpha \sin \beta t$$

Rearranging: $\sin \beta t [k_1 (k - \beta^2 m) - \alpha] + \cos \beta t [k_2 (k - \beta^2 m)] = 0$

→ so this eqn. would need to be true for all values of t , meaning the coefficients need to be zero

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- if $k - m\beta^2 = 0$, the coefficient for $\cos \beta t$ vanishes, but the coefficient for $\sin \beta t$ is non-zero
- if $k_2 = 0$, then $\cos \beta t$ is taken care of and we can therefore have $k - m\beta^2 \neq 0$
- to make the $\sin \beta t$ term vanish, we must then have

$$(k - m\beta^2)k_1 - \alpha = 0$$

$$\rightarrow k_1 = \frac{\alpha}{k - m\beta^2} = \frac{\alpha}{m} \frac{1}{\omega^2 - \beta^2}$$

$$\Rightarrow \text{thus we have } x_p(t) = \frac{\alpha/m}{\omega^2 - \beta^2} \sin \beta t$$

□ **BUT WAIT!** We concluded we couldn't have $k - m\beta^2 = 0$ above. But this is only true for our specific trial solution. Hypothetically, one could make β anything they wanted (and could thus have $k - m\beta^2 = 0$). So we need to find another possible trial solution that would indeed work in the case β was chosen such that $k - m\beta^2 = 0$ (i.e. $\beta = \omega$).

$\rightarrow$ we are going to make a guess of the form:

$$x_p(t) = k_1 t \sin \omega t + k_2 t \cos \omega t$$

the logic here is as follows:

- we want a solution that is linearly independent from x_h (note that when $k - m\beta^2 = 0$, $\beta = \omega$!!)
- multiplying by t leads us to a 'similar' solution, but one that is LI relative to x_h

→ plugging this back into our ODE, we have

$$x_p = k_1 \sin \omega_0 t + k_2 \cos \omega_0 t$$

$$\dot{x}_p = k_1 \cos \omega_0 t + \omega_0 k_1 \sin \omega_0 t + k_2 \cos \omega_0 t - \omega_0 k_2 \sin \omega_0 t$$

$$\ddot{x}_p = -\omega_0 k_1 \sin \omega_0 t + \omega_0 k_1 \cos \omega_0 t - \omega_0^2 k_1 \sin \omega_0 t - \omega_0 k_2 \sin \omega_0 t - \omega_0 k_2 \cos \omega_0 t - \omega_0^2 k_2 \cos \omega_0 t$$

$$= \sin \omega_0 t [-2\omega_0 k_2 - \omega_0^2 k_1] + \cos \omega_0 t [2\omega_0 k_1 - \omega_0^2 k_2]$$

$$m \ddot{x}_p + k x_p = \alpha \sin \omega_0 t$$

$$\rightarrow \sin \omega_0 t [-2\omega_0 m k_2 - \omega_0^2 m k_1] + \cos \omega_0 t [2\omega_0 m k_1 - \omega_0^2 m k_2] + k_1 t + k_2 t = 0$$

but $\omega_0^2 m = k$, so the terms w/ t cancel out!

$$\Rightarrow \sin \omega_0 t [-2m\omega_0 k_2 + \alpha] + \cos \omega_0 t [2m\omega_0 k_1] = 0$$

→ so this will hold provided $k_1 = 0$ and $k_2 = -\frac{\alpha}{2m\omega_0}$, leading us to the solution

$$x_p(t) = -\frac{\alpha/m}{2\omega_0} \sin(\omega_0 t) \quad \leftarrow \text{NOTE: solutions amplitudes now grow w/ time!!}$$

□ pulling this all together, we can write the general solution to the ODE

$$m \ddot{x} + kx = \alpha \sin \beta t \quad \text{as}$$

$$x(t) = \begin{cases} C_1 \sin \omega_0 t + C_2 \cos \omega_0 t + \frac{\alpha/m}{\omega_0^2 - \beta^2} \sin \beta t & [\beta \neq \omega_0 = \sqrt{k/m}] \\ C_1 \sin \omega_0 t + C_2 \cos \omega_0 t - \frac{\alpha/m}{2\omega_0} t + \cos \omega_0 t & [\beta = \omega_0] \end{cases}$$

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(cont.)

□ while the solution might seem a bit complicated, the physical intuition is actually rather straight-forward

- drive the system at its resonant freq. (i.e. $\beta = \omega_0$), it just keeps storing energy and the amplitude of the resulting oscillations just grows linearly w/ time
- drive it far away from resonance and the various sinusoidal terms will 'interfere', so you'll get some sort of oscillatory response, but the amplitude will be small
- drive it near resonance, the sinusoidal terms will be more in phase and thus add together constructively and produce a finite but larger amplitude response (this is what is commonly called **BEATING**, a phenomena important in auditory perception)

⇒ look back at your 3/26 notes to see how we derived things from a bit more general of an approach here!