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Math 250B

Cauchy-Euler Eqns. (cont.; see L.L. 8.5)

GOAL: solve ODE of the form $ax^2 y'' + bxy' + cy = 0$
 (where a, b, c - const., $y' \equiv \frac{dy}{dx}$)

IDEA: note the symmetry: the power of the x term out front is the same as the order of the derivative it is associated with

→ can we use that property to our advantage to essentially morph the eqn. into one w/ const. coefficients (i.e. one we already know how to solve!)?

□ the basic idea is to make a change of variable $x \rightarrow t$ such that

$$\frac{dy}{dt} \approx x \frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} \approx x^2 \frac{d^2y}{dx^2}$$

so to essentially 'get rid' of the non-constant coefficient terms

□ let $x = e^t$ for $x > 0$ (and conversely, $x = -e^{+t}$ for $x < 0$)

then $t = \ln(x) \rightarrow \frac{dx}{dt} = e^t = x$, $\frac{dt}{dx} = \frac{1}{x}$

⇒ so applying the chain rule: $\frac{dy}{dx} = \frac{dt}{dx} \frac{dy}{dt} = \frac{1}{x} \frac{dy}{dt}$

so $\left[\frac{dy}{dt} = x \frac{dy}{dx} \right]$ (precisely what we wanted!)

□ now what about $\frac{d^2y}{dx^2}$?

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{x} \frac{dy}{dt} \right) = -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{d}{dx} \left(\frac{dy}{dt} \right)$$

(cont.)

$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x} \frac{\partial^2 y}{\partial t^2} \frac{dt}{dx} \quad (\text{just using the chain rule again!})$$
$$= -\frac{1}{x^2} \frac{dy}{dt} + \frac{1}{x^2} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

Rearranging $\leadsto x^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} - \frac{dy}{dt}$

$\Rightarrow$ so now we have both the key pieces we need (i.e. our choice $x = e^t$ was the right one), so let's put this altogether:

$$ax^2 \frac{\partial^2 y}{\partial x^2} + bx \frac{dy}{dx} + cy$$
$$= ax^2 \left[\frac{1}{x^2} \left(\frac{\partial^2 y}{\partial t^2} - \frac{dy}{dt} \right) \right] + bx \left[\frac{1}{x} \frac{dy}{dt} \right] + cy$$
$$= \boxed{a \frac{\partial^2 y}{\partial t^2} + (b-a) \frac{dy}{dt} + cy = 0}$$

$\Rightarrow$ now a linear 2nd order ODE w/ const. coefficients, which we already know how to solve!

□ so now we can solve the transformed eqn., then resubstitute x back in for t and thus determine the general solution $y(x)$

□ NOTE: our substitution only works for $x > 0$ (so we avoid taking the log of a negative #); however if $x < 0$, we could have made the change of variables $x = -e^t$ and things would have worked out in the same fashion

ex solve $x^2 y'' + xy' - y = 0$ for $x > 0$

letting $x = e^t$, our ODE becomes

$$\frac{\partial^2 y}{\partial t^2} - y = 0 \quad \leadsto \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$$

Cont.

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□ (cont.) so $y(t) = C_1 e^t + C_2 e^{-t}$

→ but remember that we want $y(x)$; in this case the resubstitution is fairly obvious

$$x = e^t, \quad e^{-t} = (e^t)^{-1} = (x)^{-1} = \frac{1}{x}$$

$$\rightarrow y(t) = C_1 e^t + C_2 e^{-t} = C_1 x + C_2 x^{-1} = y(x)$$

∴

PROBLEMS (to solve in groups)

□ Solve $t^2 \ddot{y} + 7t \dot{y} + 25y = 0$ for $t > 0$

[Note diff. in notation!]

- let $t = e^x \leftrightarrow x = \ln t$

- ODE becomes $y'' + 6y' + 25y = 0 \rightarrow \lambda^2 + 6\lambda + 25 = 0$
 $\rightarrow \lambda = -3 \pm 4i$

- $y(x) = e^{-3x} [C_1 \cos 4x + C_2 \sin 4x]$

$$= t^{-3} [C_1 \cos(4 \ln t) + C_2 \sin(4 \ln t)] = y(t)$$

∴

[2] solve $x^2 y'' + 5xy' + 4y = 0$ for $x > 0$

let $x = e^t \leftrightarrow t = \ln x$

$\rightarrow \ddot{y} + 4\dot{y} + 4y = 0 \rightarrow \lambda^2 + 4\lambda + 4 = 0$

$\rightarrow \lambda = -2$ (a double root!)

so $y(t) = C_1 e^{-2t} + C_2 t e^{-2t}$

$= C_1 x^{-2} + C_2 x^{-2} \ln x$

$= \frac{1}{x^2} [C_1 + C_2 \ln x] = y(x) \quad \therefore$

[3] Verify that for the ODE $y'' + \frac{a^2}{x^2} y = 0$ ($x > 0$, $a = \text{const.}$) that the following are true:

[a] if $a^2 < \frac{1}{4}$, $y(x) = C_1 x^{r_+} + C_2 x^{r_-}$ where $r_{\pm} = \frac{1}{2} [1 \pm \sqrt{1 - 4a^2}]$

[b] if $a^2 = \frac{1}{4}$, $y(x) = C_1 \sqrt{x} + C_2 \sqrt{x} \ln x$

[c] if $a^2 > \frac{1}{4}$, $y(x) = \sqrt{x} [C_1 \cos(\omega \ln x) + C_2 \sin(\omega \ln x)]$
where $\omega = \frac{1}{2} \sqrt{4a^2 - 1}$

$\rightarrow$ re-express as $x^2 y'' + a^2 y = 0$, let $x = e^t \leftrightarrow t = \ln x$

ODE becomes $\ddot{y} - \dot{y} + a^2 y = 0 \rightarrow \lambda^2 - \lambda + a^2 = 0$

so $\lambda = \frac{1 \pm \sqrt{1 - 4a^2}}{2}$

and now you just need to flesh out all three possible cases

$\therefore$