

7/19/10

Math 250B

Non-Homogeneous 2nd order ODEs (i.e. 'forcing functions'; see L.L.ch.7)

□ consider the linear, non-homogeneous ODE:

$$a_2(t)\ddot{x} + a_1(t)\dot{x} + a_0(t)x = f(t)$$

NOTE: the ~~coefficients~~ coefficients here need not be const.!

ex $7\ddot{x} + 2\dot{x} = \cos(t)$

□ the Existence + Uniqueness (E+U) Theorem still applies (see 3/31/10 notes)

□ a general solution can be written as

$$x_g(t) = x_h(t) + x_p(t)$$

x_g - general solution (which we hope to find!)

x_h - solution to the homogeneous eqn. (i.e. $f(t) = 0$)

x_p - a 'particular' solution to the non-homogeneous eqn.

□ this idea generalizes to higher order linear ODEs of the form

$$\frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1(t) \frac{dx}{dt} + a_0(t)x = h(t)$$

but for simplicity/clarity, we will focus on 2nd order ODEs (w/ the knowledge that the general concepts 'scale up' for higher order eqns.)

□ so the general method for finding a solution will go as follows:

1] find two linearly independent solutions to the homogeneous version of our 2nd order ODE

2] use that to write the solution component $x_h(t)$

3] find a particular solution $x_p(t)$ to the non-homogeneous eqn.

4] combine the two in order to get the general solution: $x_g(t) = x_h(t) + x_p(t)$

5] impose the initial or boundary conditions, if any

□ for #1 above, we already know how to handle the case when we have const. coefficients, but not necessarily the case when the coefficients are not const. (we will look initially at an example of how to do this with what are called **CAUCHY-EULER EONS.**)

□ for step #3, we are also going to need a means to find a particular solution for the non-homogeneous case; this is done via the **METHOD OF UNDETERMINED COEFFICIENTS** (which work when $f(t)$ is either an exponential, polynomial, sinusoid or some combination of these three)

⇒ thus we will need to restrict ourselves to some specific forms for the ODEs we can find solutions to, the general ideas/methods can be applied to a more general sense for a particular problem at hand

Cauchy-Euler ODE (CE ODE)

□ this ODE has the following form:

$$\cancel{b_2} x^2 \frac{d^2 y}{dx^2} + b_1 x \frac{dy}{dx} + b_0 y = 0$$

→ thus it is a special type of homogeneous 2nd order ODE without const. coefficients

□ the basic idea here is to make a change of variables for x such we turn this into an ODE w/ const. coefficients; a key observation to note in the CE ODE is that the power of the coefficient is the same as the order of the derivative

→ can we use that to our advantage?

ex consider the simplest case: $x^2 y'' + x y' + y = 0$

→ try a solution of the form $y = x^r$ (where r is a const. we can figure out later)

$$y' = r x^{r-1}$$

$$y'' = r(r-1) x^{r-2}$$

plugging back into the ODE

$$\begin{aligned} & \rightarrow x^2 r(r-1) x^{r-2} + x r x^{r-1} + x^r \\ & = x^r [r(r-1) + r + 1] = 0 \end{aligned}$$

⇒ so we just need to choose r such that $[r(r-1) + r + 1] = 0$ and our solution would work; this is almost like having a characteristic eqn., so it seems we might be on the right track...

ex Consider the next simplest case: $ax^2y'' + bxy' + cy = 0$

$$\rightarrow x^r [ar(r-1) + br + c] = 0$$

ASIDE:

consider the ODE w/ const. coefficients $ay'' + (b-a)y' + cy = 0$ and assume

$$y(x) = e^{\lambda x}$$

$$\rightarrow [a\lambda^2 + (b-a)\lambda + c] e^{\lambda x}$$

$$= [a\lambda(\lambda-1) + b\lambda + c] e^{\lambda x} = 0$$

$\Rightarrow$ so the eqn. we got was equivalent to the characteristic eqn. for an ODE w/ constant coefficients!

IDEA

let's make the following change of variables:

$$x = \begin{cases} e^t & x > 0 \\ e^{-t} & x < 0 \end{cases}$$

$$\rightarrow dx = \begin{cases} e^t dt = x dt & x > 0 \\ -e^{-t} (-dt) = x dt & x < 0 \end{cases}$$

now applying the chain rule, realizing that $\frac{dt}{dx} = \frac{1}{x}$ (from above)