

4/2/10

Math 250B

- so we fleshed out a specific example which had complex eigenvalues, but what about the more general case? Consider:

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \quad \longleftrightarrow \quad \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \bar{M} \bar{x}$$

now we make the following definitions:

- $\text{tr}(M) = a + d$ (the sum of the diagonals of M , this is known as the trace)
- $\det(M) = |M| = ad - bc$ (this is just the determinant as we've already encountered before)

$\Rightarrow$ then we can reexpress our corresponding eigenvalues as

$$\lambda = \frac{\text{tr} M}{2} \pm \frac{1}{2} \sqrt{(\text{tr} M)^2 - 4(\det M)} = \lambda_{1,2}$$

Some consequences: $\text{tr} M = \lambda_1 + \lambda_2$, $\det M = \lambda_1 \lambda_2$

- using this new formulation, it becomes easy to determine the qualitative behavior of solutions and thus what the corresponding phase portrait should look like for linear autonomous systems [and thus the stability of the equilibrium solution $(0,0)$]

ex Consider $\dot{x} = 2x - 2y$
 $\dot{y} = 2x - 3y$

$M = \begin{bmatrix} 2 & -2 \\ 2 & -3 \end{bmatrix} \rightarrow \text{tr } M = -1, \det M = -2$

two, real
distinct
eigenvalues

$\rightarrow \lambda = -\frac{1}{2} \pm \frac{1}{2} \sqrt{1+8} = -\frac{1}{2} \pm \frac{3}{2} = 1, -2$

so our general solution will be (see 4/5/10 notes)

$x(t) = C_1 e^t + C_2 e^{-2t}$

$y(t) = \frac{1}{(-2)} C_1 (1-2) e^t + \frac{1}{(-2)} C_2 (-2-2) e^{-2t}$
 $= \frac{1}{2} C_1 e^t + 2 C_2 e^{-2t}$

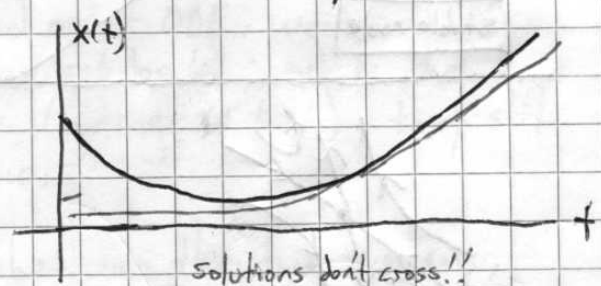
$\rightarrow$ so depending upon the initial conditions (and thus C_1 and C_2), solutions could decrease for increasing t , but ultimately will always diverge (i.e. for $t \rightarrow \infty$, $x, y \rightarrow \pm \infty$) UNLESS $C_1 = 0$

$\rightarrow$ consider the phase plane [arrows indicate what solutions do for increasing t]



these straight-line orbits are the EIGENVECTORS of the system

- most solutions diverge, but certain ones can converge to the equilibrium
- all solutions are unique
- possible solutions 'try via p plane!':



$\Rightarrow$ this case is called a **SADDLE**; the equilibrium point $(0,0)$ is potentially stable (if you start on an eigenvector, i.e. $C_1 = 0$) but is unstable otherwise