

□ another popular modeling context that utilizes a system of first order autonomous ODE in a basic **EPIDEMIC** model:
(aka SIR model)

$$S' = -\beta IS + \gamma R$$

$$I' = \beta IS - \alpha I$$

$$R' = \alpha I - \gamma R$$

where $S(t)$ - susceptible population

$I(t)$ - those infected

$R(t)$ - those recovered / 'removed'

positive
consts. $\left\{ \begin{array}{l} \beta \text{ (const.) - transmission coefficient} \\ \alpha - \text{recovery rate} \\ \gamma - \text{rate of loss of immunity} \end{array} \right.$

□ a key feature in all these models we've looked at recently is that they are **NONLINEAR**; as a result, they are hard to solve outright, but tend to exhibit rich/complex behavior

Phase Plane (see L.L. 6.1, 6.5^{6.6}) [ref. 'A Course in Math. Bio.'
SIAM]

□ when dealing w/ nonlinear systems (or linear ones for that matter!) we can get a good grasp qualitatively on the system's behavior by looking at it phase plane

- this is sometimes called a 'direction field'
- it is analogous to the phase line analysis done for single first-order ODEs
- phase plane analysis can be explored numerically via pplane

4/7/10 (cont.)

consider the autonomous system:

$$x_1' = f_1(x_1, x_2)$$

$$x_2' = f_2(x_1, x_2)$$

- at each point $x = (x_1, x_2) \in \mathbb{R}^2$, there is a vector field $f(x) = (f_1(x), f_2(x))$ representing a vector at each point in the plane laid out by x_1 and x_2
- a solution $x(t) = (x_1(t), x_2(t))$ represents a parametric curve in the plane (x_1, x_2) [this is sometimes called a trajectory or orbit]
- a tangent vector to a solution is given by $x'(t) = (x_1'(t), x_2'(t))$ and is specified by the vector field

$$f(x(t)) = (f_1(x_1(t), x_2(t)), f_2(x_1(t), x_2(t)))$$

⇒ while this might sound a bit esoteric, this is almost the same thing as the slope fields we saw last semester, except now there is essentially an extra dimension (due to the presence of another ODE!)

- sketching out some solution curves in the phase plane results in what is generally called a **phase portrait**; such is very useful for determining the overall behavior (i.e. qualitative) of solutions to a given system
 - another useful concept is that of **equilibrium points** (also called **critical points** or **fixed points**) defined as
- ⇒ $(\bar{x}_1, \bar{x}_2)$ is an equilibrium point iff $f_1(\bar{x}_1, \bar{x}_2) = f_2(\bar{x}_1, \bar{x}_2) = 0$

⇒ a quick way to get a feel for the phase portrait for a given system is to find any equilibria and determine their associated **stability** (similar to what we did last semester for a single first-order ODE)

□ the clearest way to start getting our heads around this is to initially focus on linear autonomous systems of the form:

$$\begin{aligned}x_1' &= a x_1 + b x_2 \\x_2' &= c x_1 + d x_2\end{aligned}\quad (a, b, c, d - \text{const.})$$

[of which we already know how to solve! see 4/5 notes]

ex] let's consider the following system

$$\begin{aligned}\dot{x}_1 &= \alpha x_1 + \beta x_2 \\ \dot{x}_2 &= -\beta x_1 + \alpha x_2\end{aligned}\quad \begin{array}{l} \text{where } \alpha, \beta \text{ are } \text{positive (non-zero)} \\ \text{const. } (\beta \neq 0) \end{array}$$

□ to find the equilibrium points, we set $\dot{x}_1 = \dot{x}_2 = 0$

$$\alpha x_1 + \beta x_2 = -\beta x_1 + \alpha x_2 = 0$$

$$x_1(\alpha + \beta) + x_2(\beta - \alpha) = 0$$

→ only the point $(x_1, x_2) = (0, 0)$ [i.e. the origin] satisfies the criteria and thus it is the only equilibrium point

□ we can rewrite our equation in matrix form and solve for the corresponding eigenvalues:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \leadsto \quad \bar{M} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

4/7/10 (Cont.)

→ the eigenvalues λ must satisfy the corresponding equation:

$$|\overline{M} - \lambda \overline{I}| = \begin{vmatrix} \alpha - \lambda & \beta \\ -\beta & \alpha - \lambda \end{vmatrix} = (\alpha - \lambda)^2 + \beta^2 = 0$$

[this is just the characteristic equation! rearranging ...]

$$\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2 = 0$$

$$\rightarrow \lambda = \frac{2\alpha \pm \sqrt{4\alpha^2 - 4\alpha^2 - 4\beta^2}}{2} = \alpha \pm i\beta$$

→ so our eigenvalues are complex and conjugates and provide us w/ the following solutions:

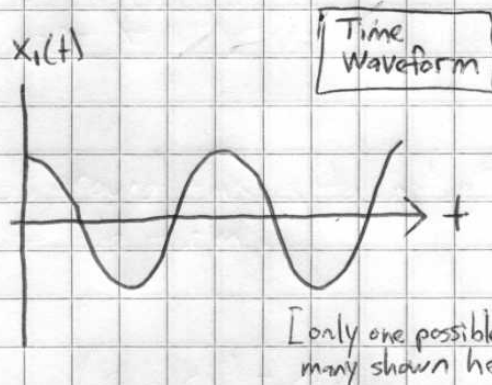
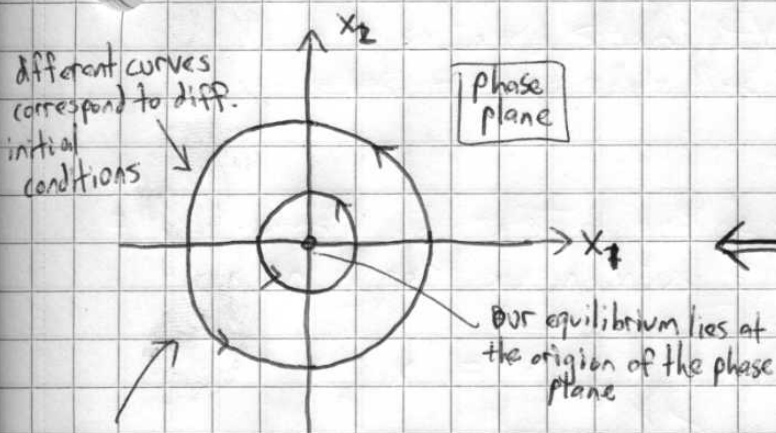
$$\begin{aligned} x_1(t) &= [C_1 \sin \beta t + C_2 \cos \beta t] e^{\alpha t} \\ &= A e^{\alpha t} \cos(\beta t + \phi) \end{aligned}$$

$$x_2(t) = -A e^{\alpha t} \sin(\beta t + \phi) \quad (\text{verify this!})$$

such that the consts. A and ϕ will be uniquely determined by the initial conditions

□ Keeping in mind that $(x_1, x_2) = (0, 0)$ is a fixed point, we can envision three different scenarios:
[all of which are oscillations in some form or another]

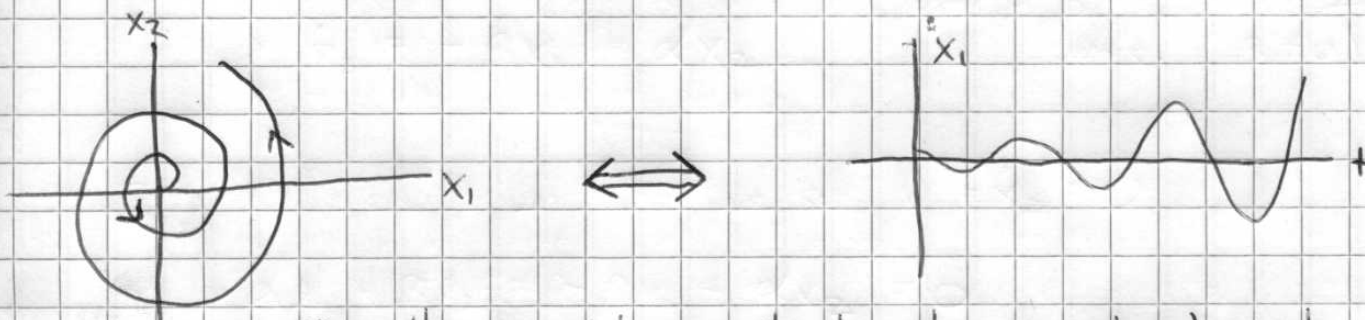
$\alpha = 0$ (eigenvalue is purely imaginary)



$\Rightarrow$ solutions are purely sinusoidal, which corresponds to a circle in the phase plane [NOTE: if you are stuck intuition-wise, think back to the harmonic oscillator - x_1 is the position of the block and x_2 is its velocity]

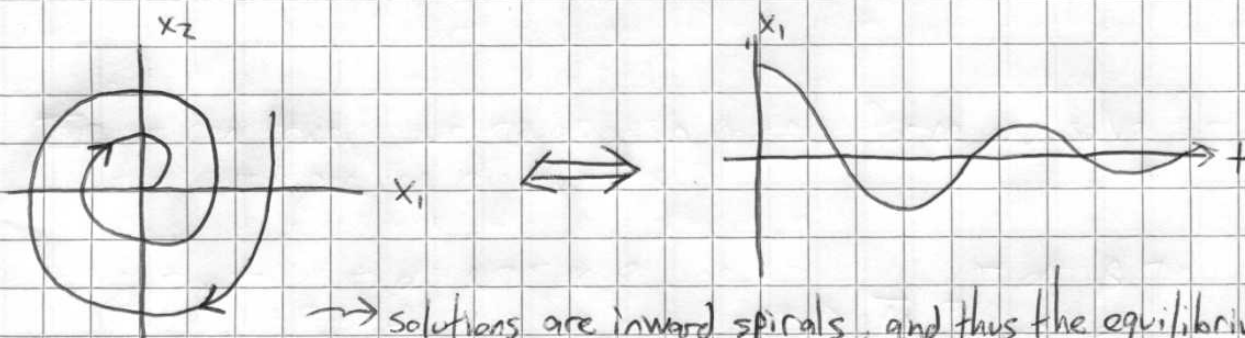
$\Rightarrow$ in this case, we classify the equilibrium point as a center

$\alpha > 0 \rightarrow$ real part is positive, so amplitude of oscillation grows for $t > 0$



$\Rightarrow$ in this case, solutions in the phase plane are outward spirals, thus the equilibrium is unstable and we classify it as a source

$\alpha < 0 \rightarrow$ real part is negative, so oscillation's amplitudes decay w/ time



$\Rightarrow$ solutions are inward spirals, and thus the equilibrium is stable and we classify it as a sink