

4/5/10

Math 250B

(ref. LL ch. 9)

our goal is to focus @ present on finding solutions to linear autonomous systems of the form:

$$x' = ax + by$$

$$y' = cx + dy$$

$$\begin{cases} x = x(t) \\ y = y(t) \\ a, b, c, d - \text{consts.} \end{cases}$$

let us differentiate the first eqn w/ respect to t :

$$\frac{d}{dt} x' = x'' = ax' + by'$$

$$= ax' + b(cx + dy)$$

[from the other eqn.]

$$= ax' + bcx + d(by)$$

$$= ax' + bcx + d(x' - ax)$$

[again, from the first eqn.]

$$= x'(a+d) + x(bc - ad) = x''$$

$$\rightarrow x'' - (a+d)x' + (ad - bc)x = 0$$

NOTE: we would have arrived at the same equation for y !

so we've rewritten our linear autonomous system as a linear, homogeneous 2nd order ODE w/ const. coefficients, which we already know the solution to! (see 3/31 notes)

characteristic equation: $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$
(assume $x(t) \sim e^{\lambda t}$)

→ which leads to the following roots:

$$\lambda = \frac{1}{2} \left[(a+d) \pm \sqrt{(a-d)^2 - 4(ad-bc)} \right]$$

$$= \frac{1}{2} \left[(a+d) \pm \sqrt{(a-d)^2 + 4bc} \right]$$

which, as we have seen before, leads to one of three cases:

□ Two real, distinct roots $[(a-d)^2 + 4bc > 0] \rightarrow$ eigenvalues are purely real

$$\lambda_1 = \frac{1}{2} [a+d + \sqrt{(a-d)^2 + 4bc}] \quad \lambda_2 = \frac{1}{2} [a+d - \sqrt{(a-d)^2 + 4bc}]$$

and $x(t)$ is just a linear combination of two exponentials:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

and from the first ODE in the system ($\dot{y} = x' - ax$), we can solve directly for $y(t)$

$$y(t) = \frac{1}{b} [x' - ax] = \frac{1}{b} [\lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} - a C_1 e^{\lambda_1 t} - a C_2 e^{\lambda_2 t}]$$

$$= \frac{1}{b} C_1 (\lambda_1 - a) e^{\lambda_1 t} + \frac{1}{b} C_2 (\lambda_2 - a) e^{\lambda_2 t} \quad \therefore$$

□ One real, repeated root $[(a-d)^2 + 4bc = 0] \rightarrow$ only one (real) eigenvalue

general solution then has the form: $x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

where we simply have $\lambda = \frac{a+d}{2}$

4/5/10 (cont.)

• and similarly as before, we can solve for $y(t)$:

$$\begin{aligned} y(t) &= \frac{1}{b} [x' - ax] = \frac{1}{b} [\lambda C_1 e^{\lambda t} + C_2 e^{\lambda t} + \lambda C_2 t e^{\lambda t} - a C_1 e^{\lambda t} - a C_2 t e^{\lambda t}] \\ &= \frac{1}{b} [C_1(1-a) + C_2] e^{\lambda t} + \frac{1}{b} C_2 (\lambda - a) t e^{\lambda t} \quad \therefore \end{aligned}$$

[3] Two distinct complex roots $[(a-d)^2 + 4bc < 0] \rightarrow$ eigenvalues have an imaginary part and are complex conjugates

solving for the resulting eigenvalues:

$$\lambda_1 = \frac{1}{2} [a+d - i\sqrt{-(a-d)^2 - 4bc}] = \alpha - i\beta$$

$$\lambda_2 = \frac{1}{2} [a+d + i\sqrt{-(a-d)^2 - 4bc}] = \alpha + i\beta \quad (\alpha, \beta \in \mathbb{R})$$

NOTE
 $\lambda_1 = \lambda_2^*$
 $\lambda_2 = \lambda_1^*$
(they are complex conjugates)

and we can express the solution to x in a # of different ways:

$$x(t) = e^{\alpha t} [C_1 \sin \beta t + C_2 \cos \beta t]$$

$$= A e^{\alpha t} \cos(\beta t + \phi)$$

$$= K e^{\lambda_1 t} + K^* e^{\lambda_2 t}$$

$$\begin{cases} C_1, C_2, A, \phi \in \mathbb{R} \\ K, K^* \in \mathbb{C} \end{cases}$$

sticking w/ the first representation and solving for y , we have:

$$y(t) = \frac{1}{b} e^{\alpha t} \{ [\alpha - a] C_1 - \beta C_2 \} \sin \beta t + \{ \beta C_1 + [\alpha - a] C_2 \} \cos \beta t$$

NOTE: solutions would be different if we originally solved for $y(t)$ and then subsequently $x(t)$, but ultimately the two approaches lead to the exact same result (as they have to!)

so now we have a general method for handling all linear autonomous systems of two coupled first order ODEs

note that when $b = c = 0$, the two equations are uncoupled (i.e. $x' = ax$ and $y' = dy$) and the solution is simply

$$\begin{aligned} x(t) &= C_1 e^{at} \\ y(t) &= C_2 e^{dt} \end{aligned}$$

a convenient way to handle first order linear autonomous systems is via MATRICES: (see L.L. 9.5)

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned} \iff \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \iff \frac{d\bar{x}}{dt} = \bar{M}\bar{x}$$

where $\bar{M} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

in this formulation, the characteristic equation is specified by finding the determinant of the following difference (see below) being set equal to zero:

$$|\bar{M} - \lambda \bar{I}| = \left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$\bar{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(this is called the identity matrix)

$$= \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

we set this determinant equal to zero

⇒ note that this is just the same (characteristic) equation we already derived!!

[for those taking linear algebra, since we have eigenvalues of a corresponding matrix, there are going to be corresponding eigenvectors as well!]