

4/2/16

Math 250B

Systems of ODEs (ref. LL ch. 6.2)

□ Consider for example the 2nd order ODE:

$$ax'' + bx' + cx = f(t)$$

now we introduce a new variable y such that $x' = y$. Then it follows that

$$y' = \frac{1}{a} [f(t) - by - cx]$$

or rewriting this out

$$ax'' + bx' + cx = f(t) \iff \begin{cases} x' = y \\ y' = \frac{1}{a} [f(t) - by - cx] \end{cases}$$

this is a 'system', two (or more) coupled
first order ODEs

⇒ thus, there is a direct, natural connection between a 2nd order ODE and a system of 1st order ODEs

DEFINITION

A system of first order ODEs has the form

$$\begin{cases} x' = P(t, x, y) \\ y' = Q(t, x, y) \end{cases}$$

where P and Q are simply functions of t, x, y only. If the system is subject to the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$, finding solutions to the system is called an **INITIAL VALUE PROBLEM**

□ Note that for the system, there will be **TWO** solutions [one for $x(t)$ and one for $y(t)$], just like how we had two solutions for a 2nd order ODE

- suppose our system had three coupled ODEs (instead of just two); how many solutions would there be? [3]
- Conversely, how many solutions (linearly-independent ones!) do you think there would be for a third order ODE? [3]

□ the **GENERAL SOLUTION** to a system contains every possible solution, though in many cases, a solution can not be found explicitly (but it can be approximated, either analytically or numerically)

i.e. think of below as definitions

□ Systems of first order ODEs can be classified depending upon the form of P and Q :

□ if $P = P(x, y)$ and $Q = Q(x, y)$ (i.e. both are independent of t), the system is said to be **AUTONOMOUS**

[NOTE: you can have a homogeneous 2nd order ODE whose corresponding system is not autonomous]

□ If P and Q are linear functions of x and y , i.e.:

$$P(x, y, t) = a(t)x + b(t)y + e(t)$$

$$Q(x, y, t) = c(t)x + d(t)y + f(t)$$

then the system is said to be **LINEAR** (if a system is not linear, it is said to be nonlinear)

⇒ if general, it is **MUCH** easier to find solutions to linear systems than nonlinear ones, but unfortunately many 'real-world' systems are nonlinear...

- For a linear system, if the coefficients are independent of t , i.e.:

$$P(x, y) = ax + by$$

$$Q(x, y) = cx + dy$$

where a, b, c, d - const.

this homogeneous system is said to have **CONSTANT COEFFICIENTS**

→ in general, these types of systems are fairly easy to solve

(damped, undriven)

- consider the harmonic oscillator $m\ddot{x} + b\dot{x} + kx = 0$

let $v = \dot{x}$ (this is just the velocity!)

→ then we can rewrite our equation as the following system:

$$\dot{x} = v$$

$$\dot{v} = -\frac{b}{m}v - \frac{k}{m}x = -\gamma v - \omega_0^2 x$$

$$\left(\begin{array}{l} \gamma \equiv \frac{b}{m} \\ \omega_0 \equiv \sqrt{\frac{k}{m}} \end{array} \right)$$

- this is an autonomous, linear system w/ const. coefficients
- Note that since we already know the solution to the 2nd order ODE, since it is equivalent to this system, then we also already know the solution to the system as well! Since we know $x(t)$, we can find $v(t)$ simply by taking a derivative.

(3/31 notes)

- just as is the case for 2nd order ODEs, we can also talk about the existence and uniqueness of solutions

THEOREM [Existence + Uniqueness for systems]

Consider the linear system of first order ODEs

$$x' = a(t)x + b(t)y + e(t)$$

$$y' = c(t)x + d(t)y + f(t)$$

subject to the initial conditions $x(t_0) = x_0$ and $y(t_0) = y_0$. If $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$ and $f(t)$ are continuous for some interval $t \in [\alpha, \beta]$ (i.e. $\alpha < t < \beta$) and t_0 is on said interval, then there exists a unique solution to the system valid for $\alpha < t < \beta$.

NOTE: There are two 'unique' initial conditions, so for a solution to truly be unique, both those conditions in turn need to be uniquely specified. Or put another way, you could have two unique solutions that both share the point $x(t_0) = x_0$ in common, but have different values for $y(t_0)$.

[think about this carefully, because now we essentially have an extra 'dimension' to things, making the picture a bit more complicated than a single first order ODE.]

- we will focus mostly on 2nd order ODEs (or, conversely, systems of two equations), but the methods we develop can readily be generalized to equations of higher order (or, again, systems with more than two dependent variables)