

Question: If the function $x(t) = A \cos t + B \sin t$ (i.e. a real-valued, oscillating function) is a solution to given 2nd order ODE, how can a complex exponential solution also work?

→ a few handy things to keep in mind:

$$\cos \theta = \frac{1}{2} [e^{i\theta} + e^{-i\theta}] , \sin \theta = \frac{1}{2i} [e^{i\theta} - e^{-i\theta}] , \frac{1}{i} = -i$$

$$\text{suppose } A \cos \theta + B \sin \theta = a e^{i\theta} + b e^{-i\theta}$$

→ find a and b in terms of A and B (and vice versa)

$$A \cos \theta + B \sin \theta = \frac{A}{2} [e^{i\theta} + e^{-i\theta}] - \frac{iB}{2} [e^{i\theta} - e^{-i\theta}]$$

$$= e^{i\theta} \left[\frac{A}{2} - i \frac{B}{2} \right] + e^{-i\theta} \left[\frac{A}{2} + i \frac{B}{2} \right] = a e^{i\theta} + b e^{-i\theta}$$

$$\Rightarrow \begin{cases} a = \frac{1}{2} [A - iB] \\ b = \frac{1}{2} [A + iB] \end{cases}$$

Note: a and b (both complex) are what is called **COMPLEX CONJUGATES** (see definition below). So there is a degree of redundancy here (if you know one, then you know the other)

Definition: Consider the complex value $z = a + ib$. Then the complex conjugate of z (denoted as z^*) is just $z^* \equiv a - ib$.

→ conversely, we could rearrange and do some algebra to show:

$$\begin{cases} A = a + b \\ B = i(a - b) \end{cases}$$

3/31/10

- goal is to now outline the general method for solving a 2nd order homogeneous ODE w/ const. coefficients of the form:

$$ax'' + bx' + cx = 0$$

a, b, c - consts.

□ trial solution: $x(t) = e^{\lambda t}$

□ plug in to get the CHARACTERISTIC EQUATION:

$$a\lambda^2 + b\lambda + c = 0$$

↪ a quadratic w/ three distinct possibilities

- a. both roots are real and distinct (call them λ_+ and λ_- , such that $\lambda_+, \lambda_- \in \mathbb{R}$)

$$\rightarrow x(t) = C_1 e^{\lambda_+ t} + C_2 e^{\lambda_- t}$$

where C_1 and C_2 are consts. that are uniquely defined by a given set of initial conditions

- b. both roots are equal (a repeated root), call it λ ($\lambda \in \mathbb{R}$)

$$\rightarrow x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t} = e^{\lambda t} (C_1 + C_2 t)$$

- c. the roots are complex conjugates of one another, i.e. $\lambda = \alpha \pm i\beta$ (where $\alpha, \beta \in \mathbb{R}$)

stemming from last pg., can also write this as

$$x(t) = k e^{\lambda_+ t} + k^* e^{\lambda_- t}$$

$$\rightarrow x(t) = e^{\alpha t} [C_1 \cos \beta t + C_2 \sin \beta t]$$

~~$= e^{\alpha t} [C_1 \cos \beta t + C_2 \sin \beta t]$~~

[see your ODE text (L.L.), ch. 6.3 (pgs. 242 ff) for a proof of the above]

□ one major point to notice is that in all possible cases for this 2nd order ODE, the general solution is a linear combination of two linearly independent functions (*)

□ note that $x = 0$ is a solution to $ax'' + bx' + cx = 0$, we call this the **TRIVIAL SOLUTION** and also note that it is an equilibrium solution (i.e. $x(t) = 0$ will be true for ALL values of t)

□ pg. 245 ff in your ODE text (L.L.) outlines some salient topics we have discussed up to this point (e.g. rewriting two sinusoids as one w/ a mag. and phase); also note the bold comment on pg. 247 about the 'only types' of solutions to 2nd order ODEs w/ const. coefficients

THEOREM - Existence + Uniqueness

Consider the 2nd order ODE

$$x'' + p(t)x' + q(t)x = f(t)$$

with the initial conditions $x(t_0) = x_0$ and $x'(t_0) = V_0$ for some $t_0 \in [\alpha, \beta]$. If $q(t)$, $p(t)$ and $f(t)$ are continuous on this interval, then there exists a unique solution to the ODE valid for $\alpha < t < \beta$.

[Alternate Version]

Express the ODE as $x'' = R(t, x, x')$

If $R(t, x, x')$ and the partial derivatives $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial x'}$ are defined and continuous in a region containing (t_0, x_0, V_0) , then there is a unique solution passing thru the point $x(t_0) = x_0$ and $x'(t_0) = V_0$.

Relevant book (LL)



Problems

- 6.2 - 2, 10, 11, 19*
- 6.3 - 1, 5, 6, 7, 9, 11
- 6.4 - 3, 7, 9