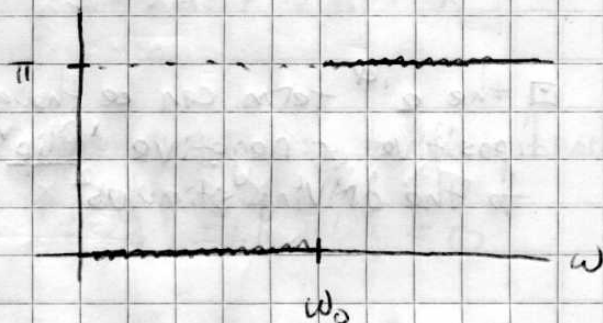
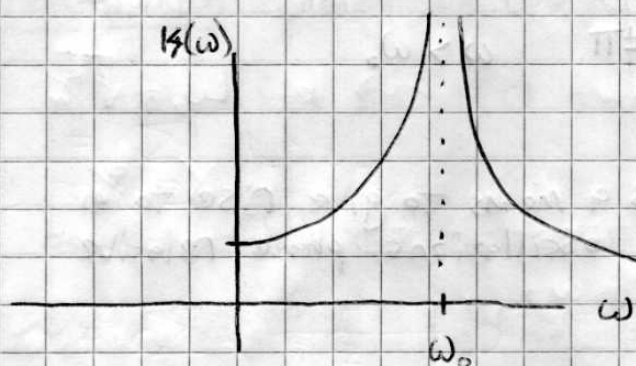


□ we can rewrite our solution a little more intuitively in terms of a **magnitude** (which is always positive) and **PHASE**

let $x(t) = K \cos(\omega t + \alpha)$ where $K = |B|$ and $\alpha = \begin{cases} 0 & \omega < \omega_0 \\ \pm \pi & \omega > \omega_0 \end{cases}$



□ we could also have found a solution using a complex exponential. For simplicity, we rewrite the ODE using a complex representation for the driving term:

$$m\ddot{x} + kx = F_0 e^{i\omega t}$$

assume solution has the form: $x(t) = A e^{i(\omega t + \phi)}$

(a 'magnitude and phase' just like before!)

NOTE: $F_0 e^{i\omega t} \neq F_0 \cos \omega t$ (actually, $F_0 \cos \omega t = \frac{F_0}{2} [e^{i\omega t} + e^{-i\omega t}]$)

NOTE: so are we taking a liberty mathematically? or simplifying our notation? [I'll leave this as an open question to think about]

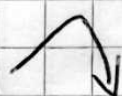
HINT: rewrite as $m\ddot{z} + kz = F_0 e^{i\omega t}$ where $x = \text{Re}(z)$

plugging in the solution's assumed form:

$$m\ddot{x} + kx = -m\omega^2 A e^{i(\omega t + \phi)} + kA e^{i(\omega t + \phi)} = F_0 e^{i\omega t}$$

$$= A e^{i\omega t} e^{i\phi} [k - m\omega^2] = F_0 e^{i\omega t}$$

$$\rightarrow A e^{i\phi} = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}$$



3/29/10 (cont.)

→ this leads us to a similar point to before:

$$A = \left| \frac{F_0/m}{\omega_0^2 - \omega^2} \right|, \quad \phi = \begin{cases} 0 & \omega < \omega_0 \\ \pm\pi & \omega > \omega_0 \end{cases}$$

□ the $e^{i\phi}$ term can be thought of as a means to give rise to a positive or negative 'flip' in the oscillation's phase relative to the driving stimulus

KEY CONCEPTS (so far)

□ Resonance - certain systems tend to oscillate, usually at a unique (or 'natural' freq.); when driven by an external force however (i.e. non-homogeneous) which is sinusoidal at said natural freq., you can end up w/ a very large response (we call this 'resonance')

□ Phase - notice that as ω goes from above to below ω_0 , the amplitude changes sign; since we are talking about sinusoidal motion, this means a phase change of $\frac{1}{2}$ cycle (or π radians), thus a convenient way to describe sinusoidal motion includes both a magnitude and phase term

$$F = m\ddot{x} = -kx + F_0 \cos(\omega t)$$

$$\rightarrow \text{assume } x(t) = A \cos(\omega t + \phi)$$

then (in terms of magnitudes): $-A m \omega^2 = -KA + F_0$
let $\phi = 0$

$\underbrace{-A m \omega^2}_{\text{mass term}}$
will dominate for large ω

$\underbrace{-KA + F_0}_{\text{stiffness term}}$
will dominate for small ω

⇒ what did we gain by using complex exponentials? Consider the more realistic case, where some degree of damping is present

DAMPING (Homogeneous Case)

$$m\ddot{x} + b\dot{x} + kx = 0$$

[b - coefficient of friction]

assume $x(t) = e^{\lambda t}$ (NOTE: if $b \neq 0$, a linear comb. of sin and cos won't work!)

$$\leadsto m\lambda^2 e^{\lambda t} + b\lambda e^{\lambda t} + k e^{\lambda t} = e^{\lambda t} (m\lambda^2 + b\lambda + k) = 0$$

$$\leadsto m\lambda^2 + b\lambda + k = 0 \quad \Rightarrow \quad \lambda = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \alpha \pm \beta i$$

A few observations

□ if $b^2 < 4mk$, $\beta \neq 0$ (i.e. $\lambda \in \mathbb{C}$), meaning solutions will oscillate

□ if $b < 0$, then since $e^{\lambda t} = e^{\alpha \pm \beta i}$, at least one solution will be an exponential to a positive power (meaning it will blow up w/ time!)

□ if $b^2 \geq 4mk$, $\beta = 0$ and solutions are simply exponentials

$$\square e^{\lambda t} = e^{\alpha \pm \beta i} = \underbrace{e^{\alpha}}_{\text{'envelope' term}} \cdot \underbrace{e^{\pm i\beta}}_{\text{oscillatory part}}$$

- $\alpha = 0$, $\beta \neq 0$ - pure oscillation
- $\alpha > 0$, $\beta \neq 0$ - growing oscillation (unstable)
- $\alpha < 0$, $\beta \neq 0$ - decaying oscillation (stable)
- $\alpha < 0$, $\beta = 0$ - decay response, no oscillation (critical damping)

DAMPING w/ SINUSOIDAL FORCING

$$F = m\ddot{x} = -kx - b\dot{x} + F_0 \cos \omega t$$

$$\rightarrow \text{let } \begin{cases} \omega_0 = \sqrt{k/m} \\ \gamma = b/m \end{cases}$$

$$\rightarrow \ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

NOTE: minus sign now why?

□ assume a solution in the simplified complex exponential form:

$$z = Ae^{i(\omega t - \delta)}$$

where $x = \text{Re}(z)$

□ plug back into the ODE (assuming the driving term to take the form $\frac{F_0}{m} \cos \omega t$):

$$-\omega^2 x + i\omega\gamma x + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t} \rightarrow Ae^{-i\delta} [\omega_0^2 - \omega^2 + i\omega\gamma] = \frac{F_0}{m}$$

$$\rightarrow Ae^{-i\delta} = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\omega\gamma} = \frac{F_0/m}{\omega_0^2 - \omega^2 + i\omega\gamma} \cdot \frac{\omega_0^2 - \omega^2 - i\omega\gamma}{\omega_0^2 - \omega^2 - i\omega\gamma}$$

$$= \frac{F_0/m}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \cdot (\omega_0^2 - \omega^2) - i\omega\gamma$$

(notice we have a polar vs. cartesian form for a complex #)

$$\Rightarrow \boxed{\begin{aligned} A(\omega) &= \frac{F_0/m}{[(\omega_0^2 - \omega^2)^2 + (\gamma\omega)^2]^{1/2}} \\ \delta(\omega) &= \frac{\gamma\omega}{\omega_0^2 - \omega^2} \end{aligned}}$$