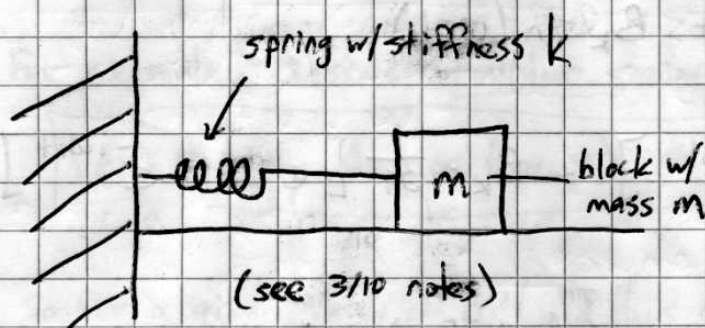


3/26/10

MATH 250B

Harmonic oscillator Revisited

[to get some further intuition re 2nd order ODEs and complex #s]



Hooke's Law: $F = -Kx$

(assume no friction for now)

Work: $W = \int_{x_i}^{x_f} F(x) dx = \int -kx dx$
(to stretch spring some distance)

$$= -\frac{1}{2} kx^2 \Big|_{x_i}^{x_f} = -\frac{1}{2} k(\Delta x)^2 \quad (\Delta x \equiv x_f - x_i)$$

→ think of as ENERGY STORED

back to Newton's 2nd Law: $F = ma = m\ddot{x} = -Kx$

→ $m\ddot{x} + Kx = 0$

[linear, homogeneous 2nd order ODE w/ const. coefficients]

SOLUTION 1: $x(t) = A_1 \sin(\omega_0 t) + A_2 \cos(\omega_0 t)$
 $= B \sin(\omega_0 t + \phi)$
 $= C \cos(\omega_0 t + \theta)$

} all these are equivalent

→ let's take the last one and plug it in:

$$x(t) = C \cos(\omega_0 t + \theta) \rightarrow \ddot{x}(t) = -\omega_0^2 C \cos(\omega_0 t + \theta)$$
$$= -\omega_0^2 x(t)$$

plugging back into the ODE:

$$-m\omega_0^2 x(t) + kx(t) = 0$$

(assuming $x(t) \neq 0$) $\rightarrow \omega_0^2 = \sqrt{k/m}$ [the 'natural' freq. the system oscillates at]

SOLUTION 2: assume $x(t) = e^{\lambda t}$ (now plug into the ODE)

$$m\lambda^2 e^{\lambda t} + ke^{\lambda t} = 0 \rightarrow \lambda = \pm\sqrt{-k/m} = \pm i\omega_0$$

so the solution (now including an arbitrary const. $\tilde{A}$ at front) has the form:

$\tilde{A} \in \mathbb{R}?$

$$x(t) = \tilde{A} e^{\pm i\omega_0 t} = \tilde{A} [\cos \omega_0 t \pm i \sin \omega_0 t]$$

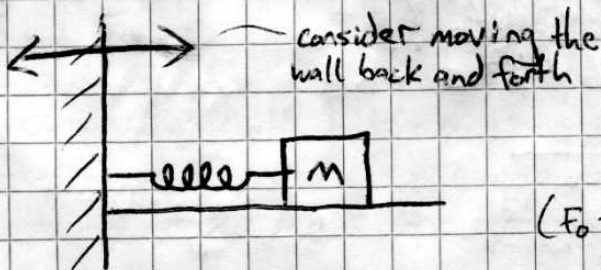
SOLUTION 3: $x(t) = A_1 \sin(\omega_0 t) + A_2 \cos(\omega_0 t)$

$\rightarrow$ plug it in, using $\omega_0 = \sqrt{k/m}$; it works

$\Rightarrow$ all these solutions have got to be equivalent, so there somehow needs to be a connection between the 'real' solutions (#1 and 3) and the complex one

[NOTE: $e^{\pm i\theta} = \cos \theta \pm i \sin \theta$!!]

□ let's now consider the non-homogeneous case such that we have a sinusoidal driving force w/ freq. ω



$$m\ddot{x} = -kx + F_0 \cos \omega t$$

(F_0 - amplitude of driving force)

3/26/10 (cont.)

→ here, the system 'wants' to oscillate with freq. ω_0 , but the driving force imposes its own freq. ω on the system

□ for now, we will consider only STEADY-STATE behavior and ignore transient effects (i.e., the system has been driven for awhile) such that the block is oscillating solely at the driving freq. ω

→ for simplicity, assume $x(t) = B \cos \omega t$
(though we could plug in a more general expression also containing $\sin \omega t$)

plugging into the ODE: $-m\omega^2 B \cos \omega t + k B \cos \omega t = F_0 \cos \omega t$

solving for B : $B = \frac{F_0}{k - m\omega^2} = \frac{F_0/m}{\omega_0^2 - \omega^2}$

→ so a solution (steady-state) to $m\ddot{x} + kx = F_0 \cos \omega t$ is

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t$$

⇒ this introduces the notion of RESONANCE: when you drive the system near its natural freq. (ω_0) via the non-homogeneous term $F_0 \cos \omega t$, your system's response can get really big!!

