

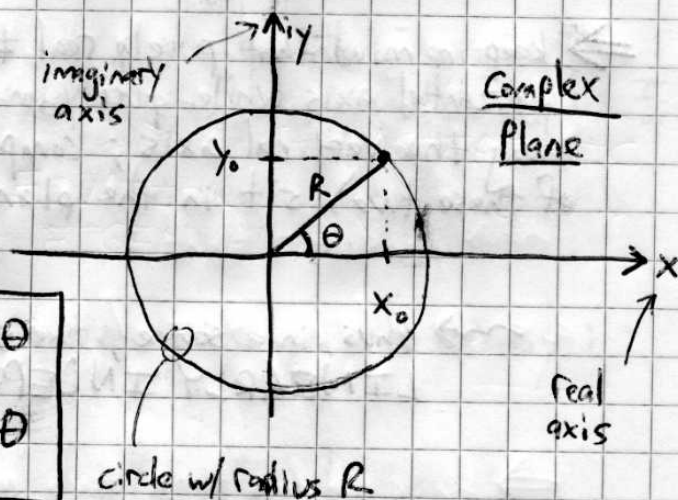
Complex #s (Review)

□ starting point - definition : $i \equiv \sqrt{-1}$

□ $\begin{array}{c} \text{Cartesian form} \\ x_0 + iy_0 \end{array} = \begin{array}{c} \text{Polar Form} \\ R e^{i\theta} \end{array}$

$x, y, R, \theta \in \mathbb{R}$ (all real #'s)

→ think geometrically as a point/circle in the complex plane (use Pythagoras' theorem to draw connections)



$$R = \sqrt{x_0^2 + y_0^2}$$

$$x_0 = R \cos \theta$$

$$\theta = \tan^{-1}\left(\frac{y_0}{x_0}\right)$$

$$y_0 = R \sin \theta$$

⇒ note that this also follows from Euler's Formula : $e^{i\theta} = \cos \theta + i \sin \theta$ (which we derived on 3/12)

□ you should feel comfortable dealing w/ complex #'s in either polar or cartesian form (and jumping back and forth between the two)

ex express the complex # $Z = 10e^{i\pi/2}$ in cartesian form

$$R = 10, \theta = \pi/2$$

$$x = R \cos \theta = 10 \cos \frac{\pi}{2} = 0$$

$$y = R \sin \theta = 10 \sin \frac{\pi}{2} = 10$$

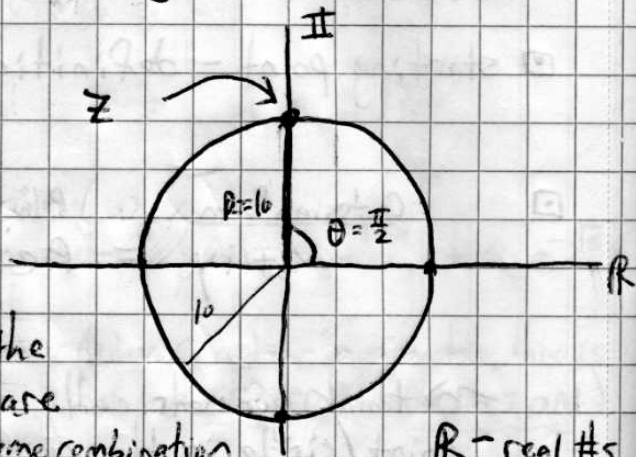
$$\rightarrow Z = 10e^{i\pi/2} = 10i$$



3/24/10 (cont.)

ex (cont.) Another way to think about this is graphically: a circle centered about the origin with radius R and the point laying on the circle at an angle θ relative to the x-axis.

→ within this context, it should seem readily apparent that the point sits at $10i$.



→ keep in mind that purely real # sit along the horizontal axis while purely imaginary #s are along the vertical axis; complex #s (some combination of these two) sit in the plane created by these two axes

R - real #s
II - imaginary #s

→ thus, in a sense, the set of real # and imaginary #s are LINEARLY INDEPENDENT

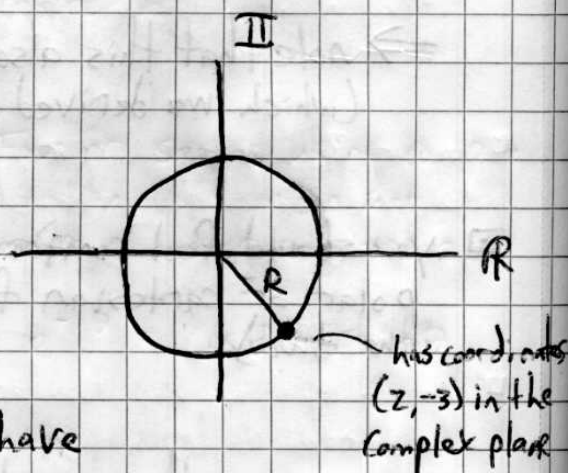
ex express $z = 2 - 3i$ in polar form

→ consider graphically in the complex plane!!

• point lies on a circle with radius $R = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

• point is in fourth quadrant, so we must have $\frac{3\pi}{2} \leq \theta \leq 2\pi$

(or conversely, $-\frac{\pi}{2} \leq \theta \leq 0$)



$$\rightarrow z = 2 - 3i = \sqrt{13} \exp[-i \arctan(3/2)]$$

(since $\arctan(-x) = -\arctan(x)$)

ex show that $\frac{1}{i} = -i$

multiply both sides by i :

$$i \left(\frac{1}{i} \right) = 1 = i \cdot -i = -i \cdot i^2 = -1 \cdot -1 = 1 \quad \checkmark$$

$\therefore$

□ sometimes very helpful to think about operations using complex #'s as geometric changes in the complex such as a rotation. For example:

$$i = 0 + 1 \cdot i \rightarrow x=0, y=1 \rightarrow R = \sqrt{0^2 + 1^2} = 1, \theta = \arctan\left(\frac{1}{0}\right) = \frac{\pi}{2}$$

$$\rightarrow \boxed{i = e^{i\pi/2}}$$

So now consider multiplying some arbitrary complex # z by i :

$$z \cdot i = R e^{i\theta} \cdot e^{i\pi/2} = R \exp[i(\theta + \frac{\pi}{2})]$$

→ multiplying by i didn't affect the magnitude, but simply **ROTATED** the point 90° (or $1/4$ cycle) in the counter-clockwise direction

□ in some engineering contexts, the symbol j is preferred for expressing current density and thus you may come across the alternate notation:

$$\boxed{j \equiv \sqrt{-1}}$$

Why are complex #'s useful?

□ Good question. Simply put, representing ~~complex~~ physical quantities as complex # facilitates book-keeping. or put another way, the exponential in the polar form makes algebraic manipulations much easier (than, say, multiplying out linear combinations of sin and cos terms)

13/24/10 (cont.)

□ contrary to what you might be told elsewhere, despite using 'imaginary' #'s, complex #'s are describing physical quantities that exist in the 'real' world (e.g. magnitude and phase of a sinusoid)

□ For example, consider Fourier series:

$$f(x) = \sum_{n=1}^{\infty} A_n \cos(\omega_n x) + B_n \sin(\omega_n x)$$

so for a given value of n (and thus a given freq. ω_n), there are essentially two free parameters to be determined, A_n and B_n (both being 'magnitudes')

→ consider that

$$a \sin \theta + b \cos \theta = A \sin(\theta + B)$$

where

$$\begin{cases} B = \arctan\left(\frac{b}{a}\right) \\ A = \frac{b}{\sin B} \\ a = A \cos B \\ b = A \sin B \end{cases}$$

⇒ thus we can turn two 'magnitudes' (a and b) into a magnitude and phase (A and B respectively)

⊗ so it becomes possible to recast our Fourier series using complex variables (this is usually how you see it in analysis or numerically on a computer, since algebraic operations are much easier to compute!)

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and thus in turn} \quad e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\begin{aligned} \cos \theta &= \frac{1}{2} [e^{i\theta} + e^{-i\theta}] \\ \sin \theta &= \frac{1}{2i} [e^{i\theta} - e^{-i\theta}] \end{aligned}$$

both these follow naturally from Euler's formula and you should be able to derive these formulae

[NOTE: You will not be tested directly on the material below on 3/24/10 this page, it is merely provided as a reference. But given that it helps tie things together, you are encouraged to think it thru!!]

[Goal] Express a Fourier series using complex variables

$$\begin{aligned} f(t) &= \sum_{k=1}^{\infty} A_k \cos(\omega_k t) + B_k \sin(\omega_k t) \\ &= \sum_{k=1}^{\infty} A_k \cdot \frac{1}{2} [e^{i\omega_k t} + e^{-i\omega_k t}] + B_k \cdot \frac{1}{2i} [e^{i\omega_k t} - e^{-i\omega_k t}] \\ &= \sum_{k=1}^{\infty} \left(\frac{A_k - iB_k}{2} \right) e^{i\omega_k t} + \left(\frac{A_k + iB_k}{2} \right) e^{-i\omega_k t} \\ &= \sum_{k=1}^{\infty} F_k e^{i\omega_k t} + F_{-k} e^{i\omega_{-k} t} \quad \left(\text{where } F_{\pm k} \equiv \frac{A_k \mp iB_k}{2}, \omega_{-k} \equiv -\omega_k \right) \\ &= F_0 + \sum_{k=1}^{\infty} F_k e^{i\omega_k t} + F_{-k} e^{i\omega_{-k} t} \quad \left(\text{where } F_0 \equiv 0, \text{ we do this for 'book-keeping'} \right) \\ &= \sum_{k=-\infty}^{\infty} F_k e^{i\omega_k t} = f(x) \quad \therefore \end{aligned}$$

$\Rightarrow$ here, the complex quantity F_k is commonly called the **FOURIER COEFFICIENT**: it has a real and imaginary part, but these two values are directly connected to the (physically real) values A_k and B_k we started with, thus we didn't make anything up, just used complex variables to 'repackage' things

□ also note that the sum now goes to $-\infty$ to ∞ (resulting in 'negative freqs. '); a lot of times the information contained in those negative freqs. is redundant (i.e. it is already contained in the positive ω_k terms) and thus okay to ignore in practical applications, hence situations where it is okay to simply 'ignore the imaginary part, because that part is not physically real' [BAH!]