

3/22/10

MATH 250B

□ consider the 2nd order ODE: $y'' + y' - 2y = 0$

□ show that $y_1(x) = Ae^x$ and $y_2(x) = Be^{-2x}$ are both solutions
(where A and B are arb. consts.)

$$\left. \begin{array}{l} y_1 = Ae^x \\ y_1' = Ae^x \\ y_1'' = Ae^x \end{array} \right\} \rightarrow y'' + y' - 2y = Ae^x + Ae^x - 2Ae^x = 0 \quad y_1(x) \text{ checks out}$$

$$\left. \begin{array}{l} y_2 = Be^{-2x} \\ y_2' = -2Be^{-2x} \\ y_2'' = 4Be^{-2x} \end{array} \right\} \rightarrow y'' + y' - 2y = 4Be^{-2x} - 2Be^{-2x} - 2Be^{-2x} = 0$$

$y_2(x)$ checks out

→ both $y_1(x)$ and $y_2(x)$ satisfy the ODE (even despite the arbitrary consts. A and B!)

NOTE: $y_1(x) = Ae^x$ and $y_2(x) = Be^{-2x}$ are **LINEARLY INDEPENDENT**

□ show that $y_3(x) = y_2(x) + y_1(x)$ is also a solution

$$\left. \begin{array}{l} y_3 = Ae^x + Be^{-2x} \\ y_3' = Ae^x - 2Be^{-2x} \\ y_3'' = Ae^x + 4Be^{-2x} \end{array} \right\} \rightarrow y'' + y' - 2y = [Ae^x + 4Be^{-2x}] + [Ae^x - 2Be^{-2x}] - 2[Ae^x + Be^{-2x}] = 0$$

$y_3(x)$ checks out

⇒ so a 2nd order ODE can have two linearly independent solutions, of which a linear combination of both is also a solution!!

□ How did we know $y_1(x)$ and $y_2(x)$ were solutions in the first place?

→ make a general guess for a solution: $y(x) = e^{\lambda x}$ (where λ is just a const.) and plug into the ODE

$$\left. \begin{array}{l} y = e^{\lambda x} \\ y' = \lambda e^{\lambda x} \\ y'' = \lambda^2 e^{\lambda x} \end{array} \right\} \rightarrow y'' + y' - 2y = \lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 2e^{\lambda x} = e^{\lambda x} (\lambda^2 + \lambda - 2) = 0$$

Since $e^{\lambda x} \neq 0$, we must have:

$$\lambda^2 + \lambda - 2 = 0$$

(this is called the **CHARACTERISTIC EQUATION**, while λ is called an **EIGENVALUE**)

→ can easily factor this:

$$\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$$

eigen $\equiv$ 'characteristic' (German)

$$\Rightarrow \lambda = -2, 1$$

• both values for λ work, thereby leading us to e^x and e^{-2x}

□ So for 2nd order, linear, homogeneous, const. coefficients ODEs, finding and solving the characteristic eqn. can lead to a pair of linearly independent solutions

□ notice that we had two arbitrary consts. before (A and B): these can be determined when suitable **INITIAL CONDITIONS** are specified

13/22/10 (cont.)

ex Find a solution to $y'' - 5y = 0$ given that $y(0) = 1, y'(0) = 0$

- assume $y(x)$ has the general form $y(x) = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} - 5e^{\lambda x} = e^{\lambda x}(\lambda^2 - 5) = 0 \rightarrow \lambda = \pm\sqrt{5}$$

- so we can write the solution as a linear combination of the two linearly independent solutions we've found:

$$y(x) = C_1 e^{\sqrt{5}x} + C_2 e^{-\sqrt{5}x}$$

(where C_1 and C_2 are consts., this is called a GENERAL SOLUTION)

- we can now 'apply' the initial conditions so to determine the consts. C_1 and C_2

$$y(0) = 1 = C_1 e^{\sqrt{5} \cdot 0} + C_2 e^{-\sqrt{5} \cdot 0} = C_1 + C_2 \rightarrow C_1 + C_2 = 1$$

$$y'(0) = 0 = \sqrt{5}C_1 e^{\sqrt{5} \cdot 0} + \sqrt{5}C_2 e^{\sqrt{5} \cdot 0} = \sqrt{5}(C_1 - C_2)$$

$$\rightarrow C_1 - C_2 = 0$$

$$[\text{two eqns. two unknowns} \Rightarrow C_1 = C_2 = \frac{1}{2}]$$

- thus we have our unique solution: $y(x) = \frac{1}{2} [e^{\sqrt{5}x} + e^{-\sqrt{5}x}]$

∴

ex Consider $y'' + 6y' + 25y = 0$. Find the general solution

→ similar to before, we can write ~~major~~ the characteristic eqn.:

$$\lambda^2 + 6\lambda + 25 = 0 \Rightarrow \lambda = \frac{-6 \pm \sqrt{36 - 100}}{2}$$

(quadratic eqn., solve for λ)

→ square root of negative #? Bad?