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Math 250B

Physical Intuition

- (1) ▢ if there is no driving force ($F(t) = 0$) and the block is at rest, it stays at rest (Newton's first law essentially)
- (2) ▢ if I push or pull the block, the spring will exert a restoring force (Hooke's law)
- (3) ▢ if I push or pull the block and then let go, it will likely start to oscillate at some frequency
- (4) ▢ if there is friction/damping, any oscillations will eventually die out

→ we can reconcile/connect our intuition both physically and mathematically via a 2nd order ODE:

Δ Newton Second Law: $F = mA$ $\Rightarrow F = m \frac{d^2x}{dt^2}$
(force = mass $\times$ acceleration)

Δ Hooke's Law: $F = -kx$

[a ^{linear} spring w/ stiffness k produces a restorative force proportional to the distance displaced from rest]

Δ Kinetic Friction: $F = -bx'$

(b - coefficient of friction)

[friction is a force acting in a direction opposite to that of an object's motion and is proportional to the object's velocity]

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→ putting all this together via Newton's third law:

$$\underbrace{m x''}_{\text{inertial force}} = \underbrace{-b x'}_{\text{friction/damping force}} - \underbrace{k x}_{\text{spring/stiffness force}} + \underbrace{F(t)}_{\text{external driving force}}$$

(NOTE: sometime we write $x' \equiv \frac{dx}{dt} \equiv \dot{x}$)

OR, rearranged: $m x'' + b x' + k x = F(t)$

→ 2nd order linear ODE w/ const. coefficients that may or may not be homogeneous depending upon the form of $F(t)$

⇒ How do you solve such an equation for $x(t)$? our physical intuition indicates what the solution likely will be (damped oscillations), but we need to flesh out a few more key mathematical pieces to get there

□ we can approach similarly as we did w/ first order equations: simply try an exponential solution of the form

$$x(t) = \tilde{A} e^{\lambda t} \quad (\text{where } \tilde{A}^{\lambda} \text{ is a const.})$$

→ and let's just consider the homogeneous case w/o damping (i.e. $b = 0$):

$$m \ddot{x} + k x = 0$$

(this is sometimes called the **SIMPLE HARMONIC OSCILLATOR**)

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$$x(t) = \tilde{A}e^{\lambda t} \rightarrow \dot{x}(t) = \lambda \tilde{A}e^{\lambda t} \rightarrow \ddot{x}(t) = \lambda^2 \tilde{A}e^{\lambda t}$$

plugging into our ODE:

$$m(\lambda^2 \tilde{A}e^{\lambda t}) + k\tilde{A}e^{\lambda t} = \tilde{A}e^{\lambda t} (m\lambda^2 + k) = 0$$

$$\leadsto m\lambda^2 + k = 0 \quad (\text{assuming } A \neq 0)$$

$$\Rightarrow \lambda = \sqrt{-k/m}$$

WAIT: square root of negative #

$\Rightarrow$ paradox?

■ so is $x(t) = Ae^{\lambda t}$ not a solution? we know (from our intuition) that solutions oscillate, so let's try a sinusoid:

$$x(t) = A \sin(\omega_0 t) \rightarrow \dot{x} = \omega_0 A \cos(\omega_0 t) \rightarrow \ddot{x}(t) = -\omega_0^2 A \sin(\omega_0 t)$$

$$-m\omega_0^2 A \sin(\omega_0 t) + kA \sin(\omega_0 t) = A \sin(\omega_0 t) [-m\omega_0^2 + k] = 0$$

$\leadsto$ so if $\omega_0 = \sqrt{k/m}$, then $x(t) = A \sin(\omega_0 t)$ satisfies $m\ddot{x} + kx = 0$

[ω_0 is called the **NATURAL FREQUENCY** of the system and indicates the frequency the system oscillates at]

- consequence $\left\{ \begin{array}{l} \bullet \text{ stiff things (large } k) \text{ oscillate at high freqs.} \\ \bullet \text{ big/massive things (large } m) \text{ oscillate at low freqs.} \end{array} \right.$

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□ similarly, $x(t) = A \cos(\omega_0 t)$ is also a solution
(where A is determined by the initial condition)

□ lastly, the linear combination $x(t) = A \sin(\omega_0 t) + B \cos(\omega_0 t)$
also satisfies $m\ddot{x} + kx = 0$

→ so is there a means to reconcile $A \sin(\omega_0 t) + B \cos(\omega_0 t)$
versus $\tilde{A} e^{i\omega_0 t}$ as solutions?

IMAGINARY #5

similarly: $i^3 = -i$
 $i^4 = 1$

let $i \equiv \sqrt{-1}$ such that $i^2 = -1$ (now bear w/ me a moment)
(consider i as just a 'special' constant)

$$\cos \theta + i \sin \theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

$$= 1 + i\theta - \frac{\theta^2}{2!} + i \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \dots$$

$$= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

$$= e^{i\theta}$$

$$\Rightarrow \boxed{e^{i\theta} = \cos \theta + i \sin \theta}$$

(Euler's Formula)

□ so back to our exponential solution:

$$\lambda = \sqrt{-k/m} = i\omega_0 \quad \Rightarrow \quad x(t) = \tilde{A} e^{i\omega_0 t} \text{ is a solution!!}$$