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Math 250B

□ a few illustrative problems relevant going into exam II that use a power series expansion to solve ODEs and integrals

Ex Find a closed form expression for the error function $\text{erf}(x)$ for x close to zero.

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \frac{u^4}{4!} + \dots$$

$$\text{let } u = -t^2$$

$$\begin{aligned} \rightarrow e^{-t^2} &= 1 + (-t^2) + \frac{(-t^2)^2}{2!} + \frac{(-t^2)^3}{3!} + \frac{(-t^2)^4}{4!} + \dots \\ &= 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} - \dots \end{aligned}$$

$$\text{So } \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}} \int_0^x \left[1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \dots \right] dt$$

$$= \frac{2}{\sqrt{\pi}} \left[t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \frac{t^9}{9 \cdot 4!} - \dots \right] \Big|_0^x$$

$$= \frac{2}{\sqrt{\pi}} \left[x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \dots \right]$$

$$= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}$$

...

- ex Solve the linear ODE $y' = 2x - y$ in two different ways subject to the initial condition $y(0) = 1$.

Method 1 (integrating factor)

• $y' + y = 2x$ $\leadsto$ has form $y' + p(x)y = q(x)$ where $\begin{cases} p(x) = 1 \\ q(x) = 2x \end{cases}$

• $\int p(x) dx = \int dx = x$ (let const. of integ. = 0) $\Rightarrow e^{\int p(x) dx} = e^x$
[this is our integrating factor]

• multiply both sides of ODE by $e^{\int p(x) dx}$:

$$e^x(y' + y) = e^x y' + e^x y = \frac{d}{dx}(ye^x)$$

$$= e^x 2x \quad (\text{other side of the ODE})$$

$$\rightarrow d(ye^x) = e^x 2x dx \quad (\text{now integrate both sides})$$

$$\int d(ye^x) = ye^x = \int e^x \cdot 2x dx \quad (\text{solve via integration by parts})$$

$$= 2[xe^x - \int e^x dx] = 2xe^x - 2e^x + c$$

$$= ye^x \quad (\text{divide both sides by } e^x)$$

$$\rightarrow y(x) = 2x - 2 + Ce^{-x}$$

• now plug in IC: $y(0) = 1 = 2(0) - 2 + Ce^0 \rightarrow c = 3$

$$\Rightarrow \boxed{y(x) = 3e^{-x} - 2 + 2x}$$

ex (cont.) Solve $y' = 2x - y$ given I.C. $y(0) = 1$

Method 2 (Power Series)

• let $y(x) = \sum_{n=0}^{\infty} a_n x^n$ (i.e. expand $y(x)$ as a Taylor series about $x=0$)
 $= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$

• then $y'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$
 $= \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$

• from our ODE then, $y' = 2x - y \iff \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 2x - \sum_{n=0}^{\infty} a_n x^n$

or, term-by-term:

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots = 2x - [a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots]$$

$$= -a_0 + x(2 - a_1) - a_2 x^2 - a_3 x^3 - \dots$$

→ this equality needs to hold for all values of x , so we can equate the coefficients term-by-term for a given power of x :

$n=0$: $a_1 = -a_0$, but since $y(x) = a_0 + a_1 x + a_2 x^2 + \dots$
 and $y(0) = 1$ (the I.C.), $a_0 = 1$

→ $a_0 = 1, a_1 = -1$

$n=1$: $2a_2 = 2 - a_1 = 2 - (-1) = 3 \rightarrow a_2 = \frac{3}{2}$

$n=2$: $3a_3 = -a_2 = -\frac{3}{2} \rightarrow a_3 = -\frac{3}{2 \cdot 3!}$

$n=3$: $4a_4 = -a_3 = -(-\frac{1}{2}) = \frac{1}{2} \rightarrow a_4 = \frac{3}{4!}$

[and you start to see the pattern]

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$$\rightarrow a_n = (-1)^n \frac{3}{n!}$$

$$\text{so } y(x) = 1 - x + \frac{3}{2!}x^2 - \frac{3}{3!}x^3 + \frac{3}{4!}x^4 - \dots$$

$$= [3 - 3x + \frac{3}{2!}x^2 - \frac{3}{3!}x^3 + \frac{3}{4!}x^4 - \dots] - 2 + 2x$$

$$= 3 \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \right] - 2 + 2x$$

$$= 3e^{-x} - 2 + 2x \quad (\text{same thing as we got via the other method!})$$

→ Key in this final part was to manipulate the expression algebraically to get it into the form consistent w/ the other solution (and realizing the Taylor series expansion for e^{-x})

so