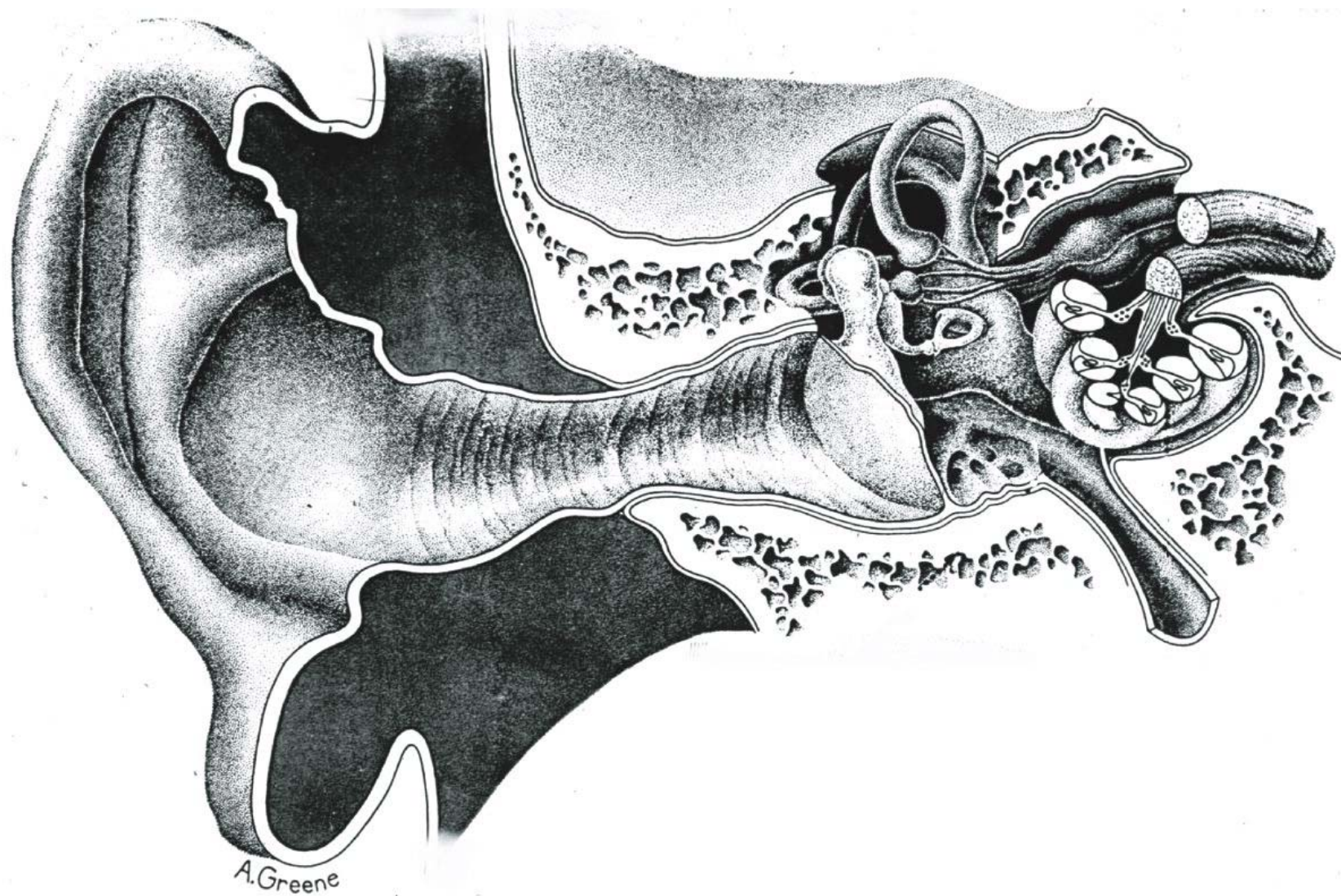




Introduction To Fourier Series

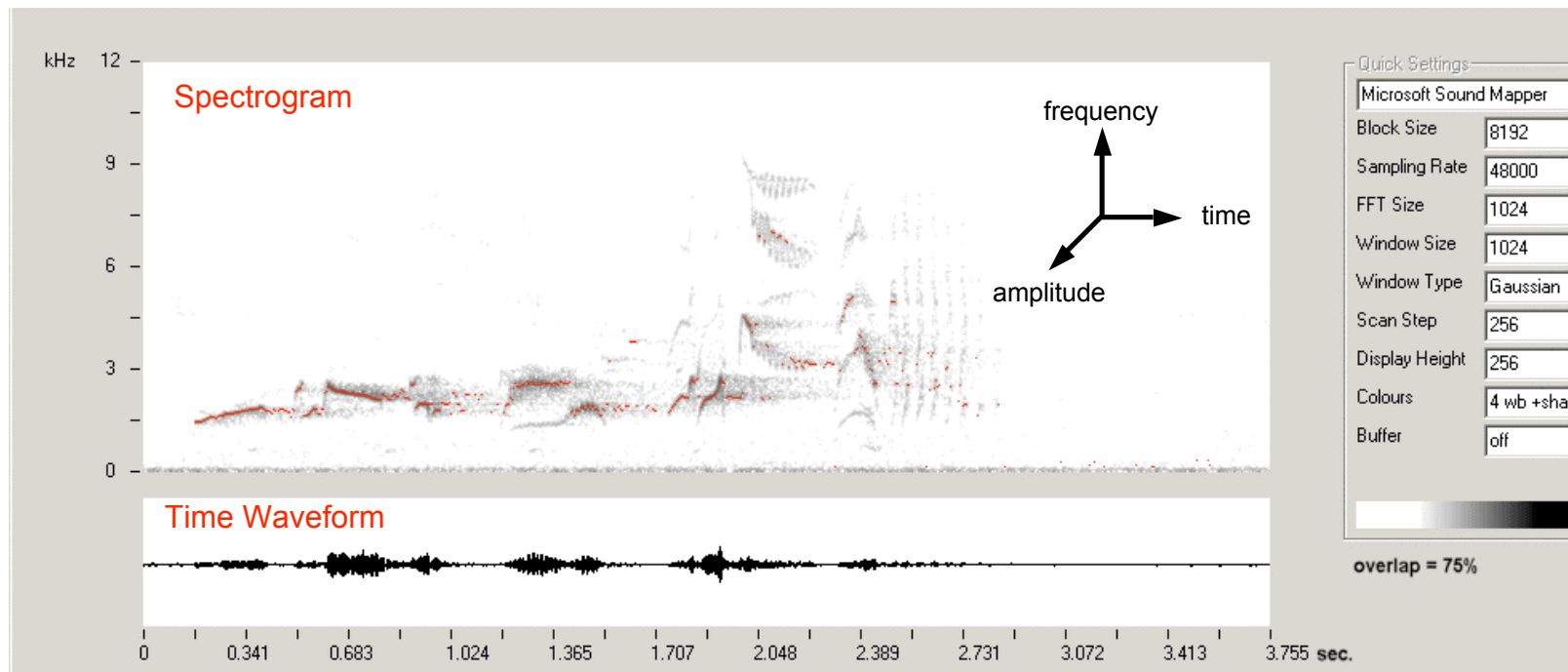
Math 250B, Spring 2010

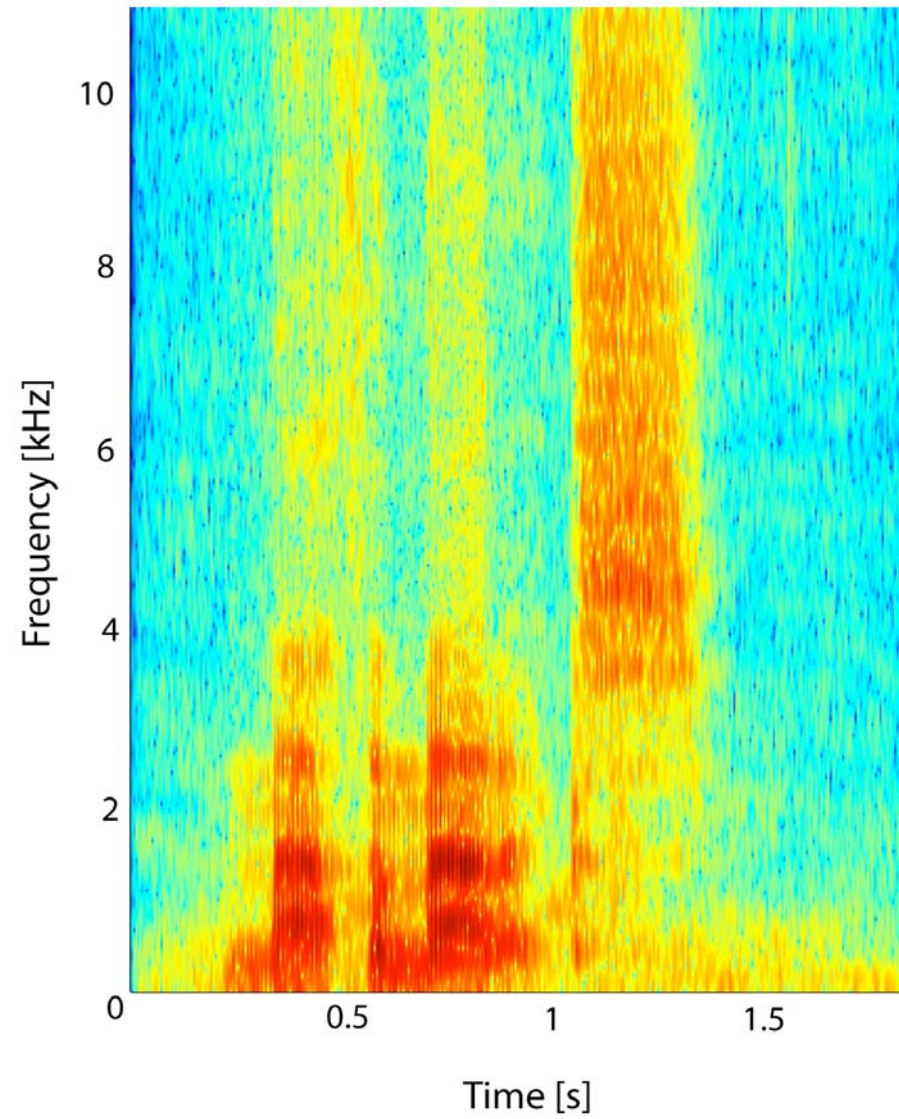
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Blackbird (*Turdus merula*)





Mathematics
analytic

Mathematics: Relationship Between Taylor and Fourier Series

Imagine a periodic time-series (w/ period 2π) described by the following function:

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2} + f^{(3)}(0)\frac{t^3}{6} + \dots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!}$$

Taylor series
about $t=0$

OR

$$\begin{aligned} \approx a_0 &+ a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \dots \\ &+ b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \dots \end{aligned}$$

Fourier series
for $t \in [-\pi, \pi]$

$$= a_0 + \sum_{n=1}^{\infty} a_k \cos nt + \sum_{n=1}^{\infty} b_k \sin nt$$

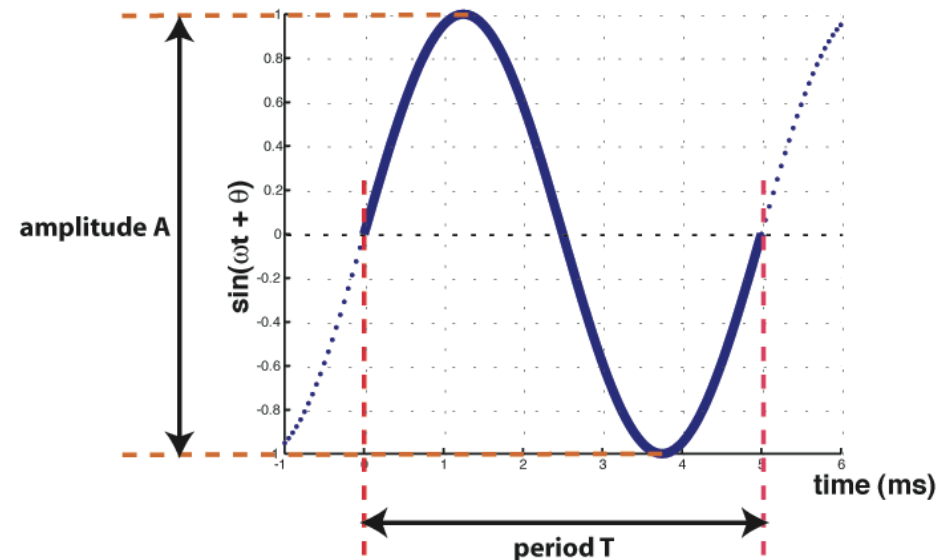
- Taylor series *expands* as a linear combination of polynomials

- Fourier series *expands* as a linear combination of sinusoids

Trigonometry review $\Rightarrow$ Sinusoids (e.g. tones)

A sinusoid has 3 basic properties:

- i. **Amplitude** - height of wave
- ii. **Frequency** = $1/T$ [Hz]
- iii. **Phase** - tells you where the peak is (needs a reference)



Why Use Fourier Series?

0. Idea put forth by Joseph Fourier (early 19'th century); his thesis committee was not impressed [though Fourier methods have revolutionized many fields of science and engineering]

1. Many phenomena in nature repeat themselves (e.g., heartbeat, songbird singing)

⇒ Might make sense to 'approximate them by periodic functions'

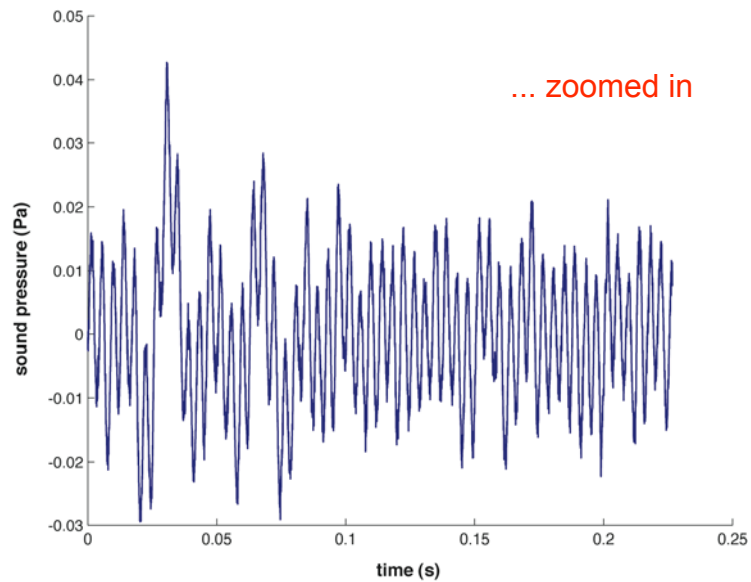
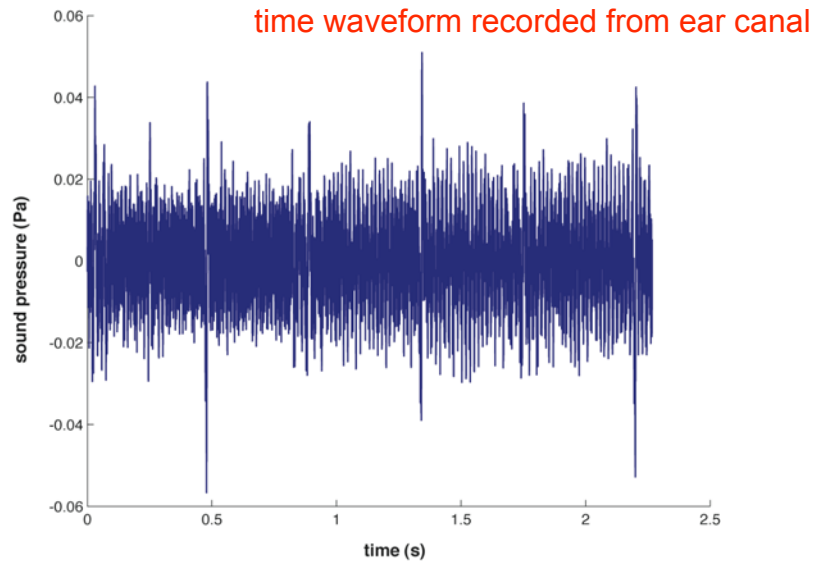
2. Taylor series can give a good *local* approximation (given you are within the radius of convergence); Fourier series give good *global* approximations

3. Still works even if $f(t)$ is not periodic

4. Fourier series gives us a means to transform from the time domain to frequency domain and vice versa (e.g., via the FFT)

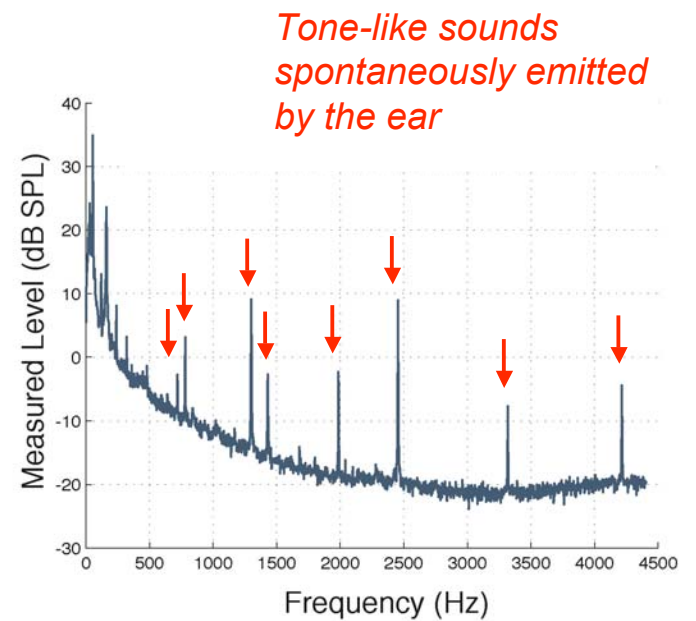
⇒ Can be easier to see things in one domain as opposed to another

Time Domain



Fourier transform

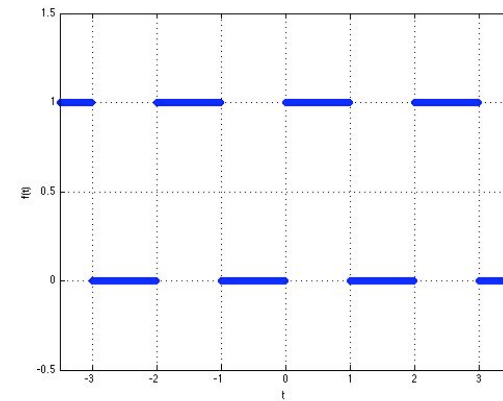
Spectral Domain



⇒ One of the ear's primary functions is to act as a Fourier 'transformer'

Example: Square Wave

$$f(t) = \begin{cases} 0 & -1 < t \leq 0 \\ 1 & 0 < t \leq 1 \end{cases}$$



For periodic function f with period b , Fourier series on $t \in [-b/2, b/2]$ is:

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos \left(\frac{2\pi kt}{b} \right) + b_k \sin \left(\frac{2\pi kt}{b} \right) \right]$$

where

$$a_k = \frac{2}{b} \int_{-b/2}^{b/2} f(t) \cos \left(\frac{2\pi kt}{b} \right) dt$$

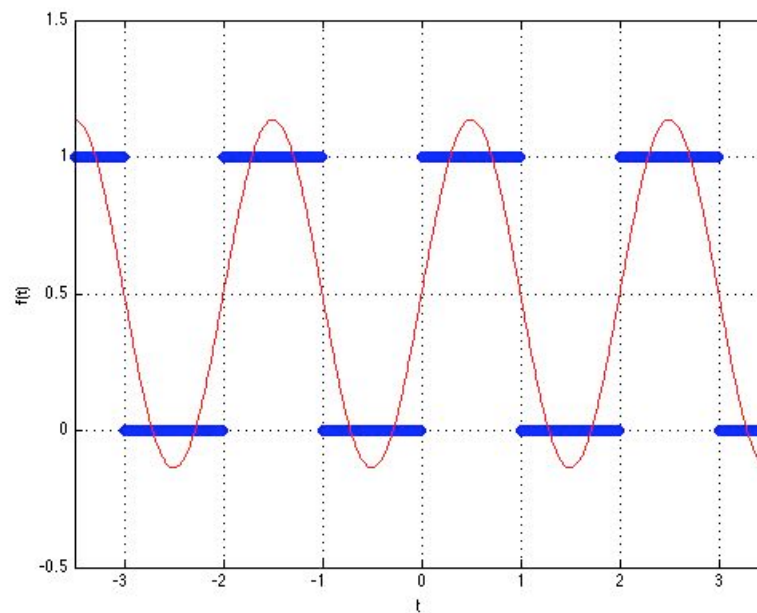
$$b_k = \frac{2}{b} \int_{-b/2}^{b/2} f(t) \sin \left(\frac{2\pi kt}{b} \right) dt$$

(these are called the *Fourier coefficients*)

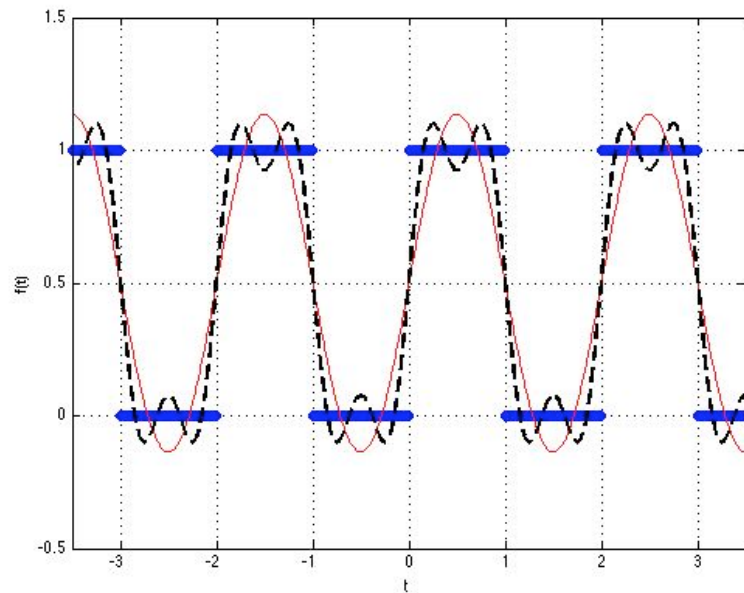
Example: Square Wave (cont.)

⇒ When the smoke clears....

$$f(t) = \frac{1}{2} + \frac{2}{\pi} \sin(\pi t) + \frac{2}{3\pi} \sin(3\pi t) + \frac{2}{5\pi} \sin(5\pi t) + \dots$$



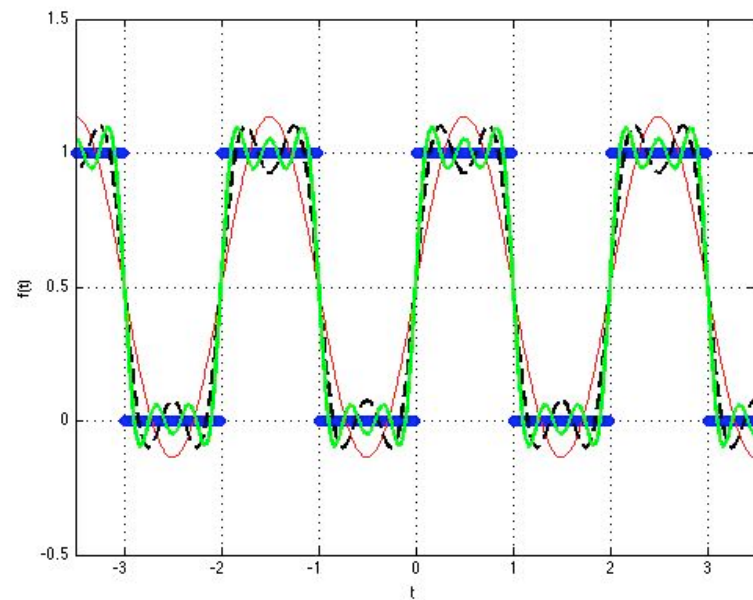
include first two terms
only (red)



include first three terms
only (black dashed)

include first four terms
only (green)

⇒ Note that approximation gets better as the
number of higher order terms included
increases



SUMMARY

- Taylor series *expands* as a linear combination of polynomials
- Fourier series *expands* as a linear combination of sinusoids
- Idea is that a function (or a time waveform) can effectively be represented as a linear combination of *basis functions*, which can be very useful in a number of different practical contexts

Fini