

3/1/10

MATH 250B

Linear Independence (cont.)

ex Consider $f(x) = \sin x$ and $g(x) = \cos x$. Is the set $\{f, g\}$ linearly independent?

$$W(f, g) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x \\ = -(\sin^2 x + \cos^2 x) = -1 \neq 0$$

$\leadsto$ since $W(f, g) \neq 0$ for all x , the set $\{f, g\}$ is linearly independent (LI) on the interval $-\infty < x < \infty$. Similarly, $\sin(mx)$ and $\sin(nx)$ [$m \neq n$] form a linearly independent set.

ex What about $f(t) = e^{-2t}$ and $g(t) = te^{-2t}$?

$$W(f, g) = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & e^{-2t} - 2te^{-2t} \end{vmatrix} \\ = e^{-2t} [e^{-2t} - 2te^{-2t} + 2te^{-2t}] = e^{-4t} \neq 0$$

$\leadsto$ thus $f(t)$ and $g(t)$ are linearly independent $\therefore$

ex Consider $\theta(x) = x^2$ and $\phi(x) = (1+x)^2$

$$W(\theta, \phi) = \begin{vmatrix} x^2 & (1+x)^2 \\ 2x & 2x+2 \end{vmatrix} = x^2(2x+2) - 2x(1+x)^2 \\ = 2x^3 + 2x^2 - 2x^3 - 4x^2 - 2x$$

(cont.)

ex (cont.) $= -2x^2 - 2x \neq 0$ for many values of x

$\Rightarrow \theta(x)$ and $\phi(x)$ are linearly independent $\therefore$

ex Consider $a(x) = e^{\lambda_1 x}$ and $b(x) = e^{\lambda_2 x}$ (where λ_1, λ_2 are constants)

$$W(a, b) = \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix}$$

$$= \lambda_2 e^{\lambda_1 x} e^{\lambda_2 x} - \lambda_1 e^{\lambda_1 x} e^{\lambda_2 x} = \lambda_2 e^{(\lambda_1 + \lambda_2)x} - \lambda_1 e^{(\lambda_1 + \lambda_2)x}$$

$$= e^{(\lambda_1 + \lambda_2)x} [\lambda_2 - \lambda_1] \neq 0 \text{ provided } \lambda_1 \neq \lambda_2$$

$\Rightarrow$ so $a(x)$ and $b(x)$ are LI if $\lambda_1 \neq \lambda_2$; but note that if $\lambda_1 = \lambda_2$, the functions will be linearly dependent (i.e. a function is linearly dependent with itself) $\therefore$

our theorem using the Wronskian can be generalized to more than two functions. For example, for n functions:

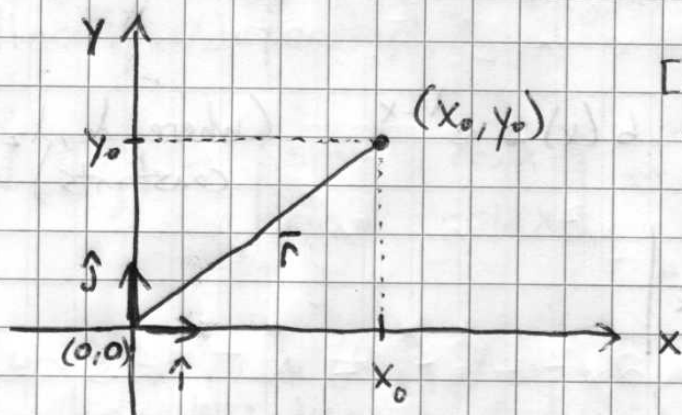
$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ f_1'' & f_2'' & \dots & f_n'' \\ \vdots & \vdots & \dots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix}$$

$\Rightarrow$ same theorem as before applies (i.e., non-zero value of W on the interval indicates LI), but determinant now tougher to compute

[this is covered more in Linear Algebra]

13/1/10 (cont.)

- A useful way to think about LI is graphically. Let's consider coordinates in the x - y plane:
[we touch upon ideas from vector calculus here just to introduce and illustrate the basic idea]



We consider here several vectors:
[a vector is just a straight line here with a certain length and direction]

$\vec{r}$ - line that extends from origin $(0,0)$ to a point (x_0, y_0)

$\hat{i}$ - line with length one that point along positive x -axis, starting at the origin

$\hat{j}$ - same as $\hat{i}$, but extends along positive y -axis

$\leadsto$ as a consequence, we have $\vec{r} = x_0 \hat{i} + y_0 \hat{j}$

$\rightarrow$ Now in the same way we defined LI for functions, we can similarly do such for two vectors. Let's assume that $\hat{i}$ and $\hat{j}$ are not LI. Then there is a c_1 and c_2 such that

$$c_1 \hat{i} + c_2 \hat{j} = \vec{0} \quad \text{with } c_1, c_2 \neq 0$$

where $\vec{0} = 0$ (i.e. a vector w/ zero length, fixed at the origin)
As a consequence, $\vec{r} + \vec{0} = \vec{r}$. Thus

$$\begin{aligned} \vec{r} + \vec{0} &= c_1 \hat{i} + c_2 \hat{j} + x_0 \hat{i} + y_0 \hat{j} = (x_0 + c_1) \hat{i} + (y_0 + c_2) \hat{j} \\ &= \vec{r} = x_0 \hat{i} + y_0 \hat{j} \end{aligned} \quad \text{[vector addition]}$$

$\leadsto$ thus $x_0 \hat{i} + y_0 \hat{j} = (x_0 + c_1) \hat{i} + (y_0 + c_2) \hat{j}$

→ if C_1 or C_2 is not zero, then there would be a single point with two sets of coordinates: (x_0, y_0) and $(x_0 + C_1, y_0 + C_2)$ [an apparent redundancy]

→ thus we need $\hat{i}$ and $\hat{j}$ to be linearly independent
[in our 2-D case here, that means the x and y axes are at right angles to one another, or what is called orthogonal]

□ coming back to functions, for Taylor series, the set of polynomials $\{1, x, x^2, x^3, \dots, x^n, \dots\}$ is linearly independent. or put another way, polynomials create a set of **BASIS FUNCTIONS** upon which we do the series expansion.

□ An expansion could also use a different set of basis functions, so long that they form a set that is linearly independent.
for example:

- Taylor series $\longleftrightarrow$ polynomials
- Fourier series $\longleftrightarrow$ sinusoids
- Laplace 'transform' $\longleftrightarrow$ exponentials [though things are a bit more complicated here...]
- wavelet transforms $\longleftrightarrow$ 'user-defined' functions
(tailored to the given application
Ⓢ hand, e.g. signal processing)