

Focus: Linear Independence

[book ref.: LL 8.4]

[Note: Even though this is an important point, we will only touch upon it lightly here; you'll definitely see it again in vector calculus (223), linear algebra (215) and other higher level mathematics courses]

- when we do a Taylor series expansion, we express a function $f(x)$ as a sum of polynomial functions, i.e.

$$\{1, x, x^2, x^3, x^4, \dots\}$$

→ why this 'family of functions'?

⇒ the answer to this question lies w/ the notion of what is called linear independence

DEFINITION (I):

A linear combination of function $(f_1, f_2, \dots, f_n)$ is defined as

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n$$

where the c_i terms are scalars (i.e. const.)
(coefficients')

- thus a Taylor series about $x=0$ can be viewed as the infinite linear combination of functions $\{1, x, x^2, x^3, \dots, x^n, \dots\}$

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DEFINITION II

A set of functions $\{f_1, f_2, \dots, f_n\}$ is linearly independent if the only linear combination equal to zero is such that all of the coefficients are equal to zero.

In other words, $\{f_1, f_2, \dots, f_n\}$ is linearly independent if

$$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0 \text{ for all } x$$

only when $c_1 = c_2 = \dots = c_n = 0$

EX consider $f(x) = x$ and $g(x) = x^2$. Is the set $\{f, g\}$ linearly independent?

let $x=1$, then $c_1(1) + c_2(1^2) = c_1 + c_2 = 0$

now let $x=2$, then $c_1(2) + c_2(2^2) = 2c_1 + 4c_2 = 0$

$\leadsto$ so both conditions lead us to $c_1 + c_2 = 2c_1 + 4c_2$
which can only be satisfied when $c_1 = c_2 = 0$

$\Rightarrow$ so the set $\{f, g\}$ is linearly independent $\therefore$

the last example might seem a bit cavalier, so we will examine a theorem which allows us a little bit more transparency in determining linear independence

DEFINITION III

The Wronskian of two differentiable function $f(x)$ and $g(x)$ is defined as

$$W(f, g) \equiv \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} \quad (\text{the determinant})$$

Theorem

Let $f(x)$ and $g(x)$ be differentiable on $x \in [a, b]$. If $W(f, g)(x_0) \neq 0$ for some $x_0 \in [a, b]$, then f and g are linearly independent on $[a, b]$. If f and g are linearly dependent, then $W(f, g) = 0$ for all x on $[a, b]$.

ex) Consider $f(x) = x$, $g(x) = x^2$ on entire x -axis

$$W(f, g) = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0 \text{ for } |x| > 0$$

$\rightarrow f(x)$ and $g(x)$ are linearly independent