

- consider the function $f(x) = (1+x)^p$ where p is a constant. Then the Taylor series about $x=0$ can be shown to be

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \dots$$

$$= \sum_{n=0}^{\infty} \binom{p}{n} x^n$$

(only valid for $-1 < x < 1$, otherwise the series diverges)

where $\binom{p}{n} \equiv \frac{p!}{n!(p-n)!}$ (' p choose n ')

⇒ this is called the binomial series with $\binom{p}{n}$ known as the binomial coefficients

(called 'binomial' because it essentially joins two terms, either 1 and x or n and p)

$$\equiv \begin{cases} 1 & \text{for } n=0 \\ \frac{p(p-1)(p-2)\dots(p-n+1)}{n!} & n \geq 1 \end{cases}$$

- an interesting mathematical aside: the binomial coefficients specify the values of Pascal's triangle (where you simply add the two #'s above)

Pascal's Triangle

			1			1			
			1		1				
			1	2	1				
			1	3	3	1			
			1	4	6	4	1		
			1	5	10	10	5	1	
			1	6	15	20	15	6	1

[and so on....]

* ⇒ the binomial series, and in particular, the binomial coefficients are used extensively in discrete mathematics, such as combinatorics (a field of study concerned with counting possible structures/combinations given a finite # of 'building blocks', useful in many engineering and biological applications), also pops up in probability and statistics

⇒ put another way, if p and n are both positive integers and $p \geq n$, then $\binom{p}{n}$ is the number of distinct sets of n objects that can be chosen from p distinct objects

2/23/10 (cont.)

ex determine $\binom{5}{2}$

[phrased another way: how many distinct pairs can I pick out of a collection of five distinct objects?]

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot (3 \cdot 2 \cdot 1)} = \frac{5 \cdot 4}{2} = 10$$

ex determine $\binom{4/3}{5}$

[p need not be an integer!]

though this starts to get into the realm of what are called gamma functions

$$\binom{4/3}{5} = \frac{(\frac{4}{3})(\frac{4}{3}-1)(\frac{4}{3}-2)(\frac{4}{3}-3)(\frac{4}{3}-4)}{5!}$$

$$= \frac{(\frac{4}{3})(\frac{1}{3})(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = -\frac{8}{3^6} = -\frac{8}{729}$$

ex determine the binomial series for $(1+x)^2$

$$(1+x)^2 = \sum_{n=0}^{\infty} \binom{2}{n} x^n$$

$$\binom{2}{0} = 1$$

$$\binom{2}{1} = \frac{2(2-1)}{1!} = 2$$

$$= 1 + 2(x) + 1(x^2) + (0)x^3 + \dots$$

$$\binom{2}{2} = \frac{2!}{2!(2-2)!} = 1$$

$$= x^2 + 2x + 1$$

$$\binom{2}{3} = 0$$

$$= (x+1)^2$$

$$\text{since } \binom{2}{3} = \frac{2(2-1)(2-3+1)}{3!} = 0 \quad \binom{2}{n} = 0 \text{ for } n > 2$$

→ just as we would expect (the Taylor series for a polynomial is just that polynomial!); keep in mind that the 'binomial series' is a specific type of Taylor series

ex Find a Taylor series about $\theta=0$ for $f(\theta) = \theta^3 \cos(\theta^2)$

→ taking numerous derivatives is hard!! try a 'substitution':

$$\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \dots$$

$$\text{so } \cos \theta^2 = 1 - \frac{(\theta^2)^2}{2!} + \frac{(\theta^2)^4}{4!} - \frac{(\theta^2)^6}{6!} + \dots$$

$$= 1 - \frac{\theta^4}{2!} + \frac{\theta^8}{4!} - \frac{\theta^{12}}{6!} + \dots$$

$$\text{so } \theta^3 \cos(\theta^2) = \theta^3 \left[1 - \frac{\theta^4}{2!} + \frac{\theta^8}{4!} - \frac{\theta^{12}}{6!} + \dots \right]$$

$$= \theta^3 - \frac{\theta^7}{2!} + \frac{\theta^{11}}{4!} - \frac{\theta^{15}}{6!} + \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{\theta^{4n+3}}{(2n)!}$$

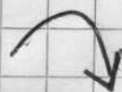
ex which function is smallest/largest for positive θ near zero?

$$1 + \sin \theta, \quad \cos \theta, \quad \frac{1}{1 - \theta^2}$$

$$\bullet 1 + \sin \theta = 1 + \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \approx 1 + \theta$$

$$\bullet \frac{1}{1 - \theta^2} = 1 + (\theta^2) + (\theta^2)^2 + (\theta^2)^3 + \dots$$

$$= 1 + \theta^2 + \theta^4 + \theta^6 + \dots \approx 1 + \theta^2$$



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ex (cont.)

$$\bullet \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \approx 1 - \frac{\theta^2}{2}$$

→ so for small values of θ ($\theta > 0$), the series expansions allow us to approximate that

$$1 + \theta > 1 + \theta^2 > 1 - \frac{\theta^2}{2}$$

(since $\theta > \theta^2$ for $0 < \theta < 1$)

$$\text{and therefore } 1 + \sin \theta > \frac{1}{1 - \theta^2} > \cos \theta$$

∴

ex (10.3.36) Stemming from a test of Einstein's theory of general relativity by looking at the bending of light around stars by a certain angle (ϕ), one needs to solve the following equation for ϕ :

$$\sin \phi + b [1 + \cos^2 \phi + \cos \phi] = 0 \quad (\text{where } b \text{ is a small positive const.})$$

→ HARD!! However if ϕ is very small (i.e. close to zero):
(expand as a Taylor series)

$$(\phi - \frac{\phi^3}{3!} + \dots) + b [1 + (1 - \frac{\phi^2}{2!} + \dots)^2 + (1 - \frac{\phi^2}{2!} + \dots)] = 0$$

$$\approx \phi + b [1 + (1 - \phi^2) + (1 - \frac{\phi^2}{2})] \quad (\text{since } \phi \text{ is very small})$$

$$\approx \phi + b [1 + 1 + 1] = \phi + 3b = 0$$

$$\Rightarrow \phi = -3b$$

→ Taylor series 'approximations' can be very useful!!

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