

12/23/10

## Tuesday Afternoon Problems

ex Airy's equation is a 2nd order ODE w/ the form

$$y'' = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}(y') = xy$$

Suppose you are given the I.C.'s  $y(0) = 1$  and  $y'(0) = 0$ .  
Using a power series expansion, find a unique solution  $y(x)$ .

$$\text{let } y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$$

$$\text{then } y'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots$$

$$\text{and } y''(x) = 2a_2 + 2 \cdot 3 \cdot a_3x + 3 \cdot 4 \cdot a_4x^2 + \dots$$

$$\text{but } y'' = xy$$

$$\begin{aligned} \rightarrow 2a_2 + 2 \cdot 3 \cdot a_3x + 3 \cdot 4 \cdot a_4x^2 + \dots &= x(a_0 + a_1x + a_2x^2 + a_3x^3 + \dots) \\ &= a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \dots \end{aligned}$$

this equality has to hold for any value of  $x$ , so we can simply equate coefficients for similar powers of  $x$ :

$$2a_2 = 0 \quad \rightarrow \quad a_2 = 0$$

$$2 \cdot 3 \cdot a_3 = a_0 \quad \text{but since } y(0) = 1, a_0 = 1 \quad \rightarrow \quad a_3 = \frac{1}{2 \cdot 3}$$

$$3 \cdot 4 \cdot a_4 = a_1 \quad \text{but since } y'(0) = 0, a_1 = 0 \quad \rightarrow \quad a_4 = 0$$

$$4 \cdot 5 \cdot a_5 = a_2 \quad \rightarrow \quad a_5 = 0$$

$$5 \cdot 6 \cdot a_6 = a_3 \quad \rightarrow \quad a_6 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}$$

Similarly,  $a_7 = a_8 = 0$ , but  $a_9 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}$  and so on...

$\Rightarrow$  putting this altogether, we have

$$y(x) = 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \dots$$

as a solution to  $y'' = xy$  with initial conditions  $y(0) = 1, y'(0) = 0$

ex determine  $\int \frac{\cos(ax)}{x} dx$  where  $a$  is a const.

$$\cos(ax) = 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} - \frac{(ax)^6}{6!} + \dots$$

$$\text{So } \int \frac{\cos(ax)}{x} dx = \int \frac{1}{x} \left[ 1 - \frac{(ax)^2}{2!} + \frac{(ax)^4}{4!} - \frac{(ax)^6}{6!} + \dots \right] dx$$

$$= \int \left[ \frac{1}{x} - \frac{a^2 x}{2!} + \frac{a^4 x^3}{4!} - \frac{a^6 x^5}{6!} + \dots \right] dx$$

$$= \ln|x| - \frac{a^2 x^2}{2 \cdot 2!} + \frac{a^4 x^4}{4 \cdot 4!} - \frac{a^6 x^6}{6 \cdot 6!} + \dots$$

$$= \ln|x| - \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} - \frac{(ax)^6}{6 \cdot 6!} + \dots$$

$$= \ln|x| + \sum_{n=1}^{\infty} (-1)^n \frac{(ax)^{2n}}{2n \cdot (2n)!}$$

2/23/10

QX show that  $e^{\sin x} = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \dots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\begin{aligned} \text{so } e^{\sin x} &= 1 + \left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right] + \frac{\left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]^2}{2!} \\ &+ \frac{\left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]^3}{3!} + \frac{\left[ x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right]^4}{4!} + \dots \end{aligned}$$

[now collect lower order terms upon multiplying things out]

$$\begin{aligned} &= 1 + x + \frac{x^2}{2!} + \left[ -\frac{x^3}{3!} + \frac{x^3}{3!} \right] + \left[ -\frac{x^4}{2 \cdot 3!} - \frac{x^4}{2 \cdot 3!} + \frac{x^4}{4!} \right] \\ &+ \left[ \frac{x^5}{5!} + \dots \right] \end{aligned}$$

[you can see how complicated things quickly get!]

$$= 1 + x + \frac{x^2}{2!} - \frac{x^4}{8} - \dots$$

↪ likely won't be able to write in a nice closed sigma form!

⇒ similarly, one can show

$$e^{\cos x} = e \left[ 1 - \frac{x^2}{2} + \frac{x^4}{6} - \frac{3!x^6}{720} + \dots \right]$$