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MATH 250B

Goal: explore some useful extensions/applications of Taylor Series
[ref. Edelstein-Keshet]

ex Evaluate $\int_0^1 \sin(x^2) dx$ analytically

▫ this integral is hard (if not impossible) to solve directly, i.e. what is the anti-derivative of $\sin(x^2)$?

▫ however, expanding the integrand via a Taylor series makes this problem much more tractable:

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!}$$

let $u = x^2$

$$\leadsto \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \dots$$

$$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}$$

$$\leadsto \int_0^1 \sin(x^2) dx = \int_0^1 \left[x^2 - \frac{x^6}{6!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \dots \right] dx$$

$$= \left[\frac{x^3}{3} - \frac{x^7}{7 \cdot 3!} + \frac{x^{11}}{11 \cdot 5!} - \frac{x^{15}}{15 \cdot 7!} + \dots \right] \Big|_0^1$$

$$= \frac{1}{3} - \frac{1}{7 \cdot 3!} + \frac{1}{11 \cdot 5!} - \frac{1}{15 \cdot 7!} + \dots \quad \text{(this series will converge quickly)}$$

$$\approx 0.31026$$

* [try putting this integral into your Riemann sums Matlab code from last semester; for example you should get a value of ≈ 0.310277 using the trapezoid rule w/ 100 steps]

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ex (highly related to the last problem)

A table of integrals indicates that

$$\int \frac{\sin(ax)}{x} dx = ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \dots$$

⇒ How did they obtain such an answer?

first, note that $\sin(ax) = ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} - \frac{(ax)^7}{7!} + \dots$

so $\int \frac{\sin(ax)}{x} dx = \int \frac{1}{x} \left[ax - \frac{(ax)^3}{3!} + \frac{(ax)^5}{5!} - \dots \right] dx$

$$= \int \left[a - \frac{a^3 x^2}{3!} + \frac{a^5 x^4}{5!} - \dots \right] dx$$

$$= ax - \frac{a^3 x^3}{3 \cdot 3!} + \frac{a^5 x^5}{5 \cdot 5!} - \dots$$

$$= ax - \frac{(ax)^3}{3 \cdot 3!} + \frac{(ax)^5}{5 \cdot 5!} - \dots = \sum_{n=0}^{\infty} (-1)^n \frac{(ax)^{2n+1}}{(2n+1)(2n+1)!}$$

* → note that to find an exact solution to this integral, we had to resort to a Taylor series; or put another way, the anti-derivative of $f(x) = \frac{\sin(ax)}{x}$ can only be expressed via an infinite series
[this ends up being true for a lot of different types of functions]

∴

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- we can in turn use Taylor series to find solutions to ODEs; we'll start by illustrating such via a simple ODE we already know the answer to

ex Find a solution to $\frac{dy}{dx} = y$ given the initial condition $y(0) = 1$

- let's start by expressing the solution (i.e. $y(x)$) via a power series:

$$y(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

- now differentiating, we have:

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

- but the ODE tells us $\frac{dy}{dx} = y$, so:

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

→ this equality must hold for all values of x , therefore the various coefficients must match:

$$a_1 = a_0, \text{ but } y(0) = 1 \Rightarrow a_0 = a_1 = 1$$

$$2a_2 = a_1, \text{ so } a_2 = \frac{a_1}{2} \Rightarrow a_2 = \frac{1}{2}$$

$$3a_3 = a_2, \text{ so } a_3 = \frac{a_2}{3} \Rightarrow a_3 = \frac{1}{6} = \frac{1}{3!}$$

$$4a_4 = a_3, \text{ so } a_4 = \frac{a_3}{4} \Rightarrow a_4 = \frac{1}{24} = \frac{1}{4!}$$

$$\text{and so on... [thereby } a_{n+1} = \frac{1}{(n+1)!}]$$

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so $y(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

but we already know that this is the Taylor series expansion for e^x !

$$\Rightarrow y(x) = e^x$$

...

ies:

up to this point, we have considered Taylor series expansions for a function $f(x)$ about $x=0$. However, we can generalize this such that the Taylor series expansion occurs about an arbitrary point $x=a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

+ ...

Ex Find the Taylor series for $\frac{1}{x}$ at $a=1$

$$f(x) = \frac{1}{x} = x^{-1}$$

$$f(1) = 1$$

$$f'(x) = -x^{-2}$$

$$f'(1) = -1$$

→

$$f''(x) = 2x^{-3}$$

$$f''(1) = 2 = 2!$$

$$f'''(x) = -3 \cdot 2 \cdot x^{-4}$$

$$f'''(1) = -3 \cdot 2 = -3!$$

in general, $f^{(n)}(1) = (-1)^n n!$

$$\Rightarrow f(x) = 1 - (x-1) + \frac{2(x-1)^2}{2!} - \frac{3!(x-1)^3}{3!} + \frac{4!(x-1)^4}{4!} + \dots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4 - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n (x-1)^n = \sum_{n=0}^{\infty} (1-x)^n$$

...