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Math 250B

ex find the Taylor series expansion for $f(x) = ax^2 + bx + c$

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

$$f(0) = c \rightarrow a_0 = c$$

$$f'(0) = b \rightarrow a_1 = b$$

$$f''(0) = \frac{2a}{2} \rightarrow a_2 = a$$

$$f'''(0) = 0 \rightarrow a_3 = 0$$

[true for all higher derivatives]

$$\rightarrow f(x) = c + bx + ax^2$$

$\Rightarrow$ the Taylor series expansion for a polynomial is just that polynomial (which should make sense given that the Taylor series is 'expanding' the function via a sum of polynomials!!)

ex Find the Taylor series expansion for $f(x) = \sin(x)$

• again we will have $\sin(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} x^n = f(x)$

$\rightarrow$ so we will need to find successive derivatives of $f(x) = \sin(x)$ and evaluate those at $x=0$ to determine the coefficients for each term

$$\begin{aligned} f(x) &= \sin x \\ f'(x) &= \cos x \\ f''(x) &= -\sin x \\ f'''(x) &= -\cos x \\ f^{(4)}(x) &= \sin x \\ f^{(5)}(x) &= \cos x \\ &\vdots \\ &\vdots \end{aligned}$$



$$\begin{aligned} f(0) &= 0 \\ f'(0) &= 1 \\ f''(0) &= 0 \\ f'''(0) &= -1 \\ f^{(4)}(0) &= 0 \\ f^{(5)}(0) &= 1 \\ &\vdots \\ &\vdots \end{aligned}$$

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$$\Rightarrow \text{so } \sin(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

therefore $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = -\frac{1}{3!}, a_4 = 0, a_5 = \frac{1}{5!}$

$$\Rightarrow \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

∴

□ so one might ask: for what values of x is this series guaranteed to converge? [i.e. what is R for this power/Taylor series?]

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \right|}{\left| \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|} = \frac{(2n+1)!}{(2n+3)!} |x^2| = \frac{x^2}{(2n+3)(2n+2)}$$

So as long as $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} < 1$, the series will converge. We can see that for our series, this will be true no matter what x is. Or put another way:

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} = 0 = k \rightarrow R = \infty$$

$\Rightarrow$ so our Taylor series expansion will converge for any value of x (for $\sin x$)

□ another way to think about the Taylor series is a means of approximating a function if you just consider the first N terms of the series (we call this the **TAYLOR POLYNOMIAL**)

$$T_N(x) = \sum_{n=0}^M a_n x^n \quad \text{where } M \geq N \text{ such that there are } N \text{ non-zero terms}$$

NOTE this notation (T_N , consistent w/ the figure on the next page), is a bit different from that used in the book for $P_n(x)$

⊛ (which extends out to the n 'th power of x)

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for example, let's graphically consider how the various Taylor polynomials approximate $f(x) = \sin(x)$

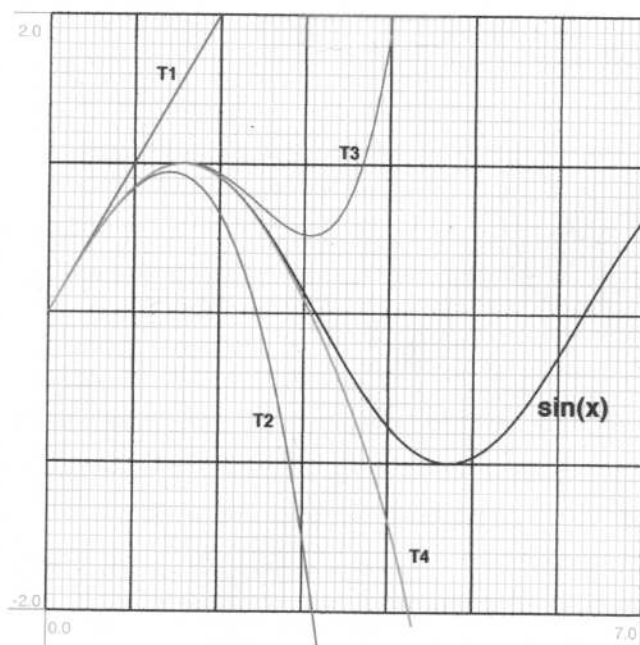


Figure 10.5. An approximation of the function $y = \sin(x)$ by successive Taylor polynomials, T_1, T_2, T_3, T_4 . The higher Taylor polynomials do a better job of approximating the function on a larger interval about $x = 0$.

[Edelstein-Keschet]

where $T_1(x) = x$ (linear)

$T_2(x) = x - \frac{x^3}{3!}$ (cubic)

$T_3(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ (quintic)

$T_4(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ (??)

Successively
higher
order
Taylor
Polynomial

⇒ notice that as you include higher and higher order terms, you can go farther out along x (i.e. away from $x=0$) before $f(x)$ and the Taylor polynomial start to diverge

⊛ ⇒ given that $R = \infty$ for the Taylor series for $f(x) = \sin x$, make sure to try to connect back to the notion of including 'all' terms

□ Similarly, the Taylor Series expansion of $f(x) = \cos(x)$ is given by

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^{2n}}{(2n)!}$$