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MATH 250B

(see Edelman-Kostant ch. 10 notes)

- let's step back for a moment; we introduced the notion of a series within the context that it could essentially represent a function, for example: (see 2/10 notes)

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{n=0}^{\infty} x^n \quad (\text{for } |x| < 1)$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} \quad (\text{for } |x| < 1)$$

→ we are going to build upon this notion of expressing a function via an infinite series for any general function $f(x)$; we call these series expansions **TAYLOR SERIES**

- suppose we have a function we wish to represent via a power series:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots = \sum_{n=0}^{\infty} a_n x^n$$

→ we can now use the function $f(x)$ directly to determine the various coefficients a_n

- consider $f(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + \dots = a_0$

$$\rightarrow a_0 = f(0)$$

- now if we differentiate $f(x)$, we have:

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots$$

$$\text{then } f'(0) = a_1 + 2a_2(0) + 3a_3(0)^2 + \dots = a_1$$

$$\rightarrow a_1 = f'(0)$$

• differentiating again, we have:

$$f''(x) = 2a_2 + 2 \cdot 3 a_3 x + 3 \cdot 4 a_4 x^2 + 4 \cdot 5 a_5 x^3 + \dots$$

$$\text{then } f''(0) = 2a_2 + 2 \cdot 3 a_3 (0) + \dots = 2a_2$$

$$\rightarrow a_2 = \frac{f''(0)}{2}$$

• we could keep doing this, on and on...

$$f'''(x) = 2 \cdot 3 a_3 + 2 \cdot 3 \cdot 4 a_4 x + 3 \cdot 4 \cdot 5 a_5 x^2 + \dots$$

$$\text{then } f'''(0) = 2 \cdot 3 a_3 + 2 \cdot 3 \cdot 4 a_4 (0) + \dots = 3! a_3$$

$$\rightarrow a_3 = \frac{f'''(0)}{3!}$$

• generalizing for the k 'th derivative:

$$f^{(k)}(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \dots \cdot k a_k + \dots$$

$$\text{then } f^{(k)}(0) = k! a_k \quad \rightarrow \quad a_k = \frac{f^{(k)}(0)}{k!}$$

$\Rightarrow$ we therefore have a 'recipe' for determining all the coefficients a_k (assuming we can compute the various derivatives of $f(x)$), allowing us to determine the Taylor series expansion of $f(x)$

NOTE: a Taylor series is a special type of power series, used to represent a function via the infinite series

ex find the Taylor series expansion for $f(x) = e^x$

• note that all derivatives of e^x are e^x , therefore

$$f^{(k)}(x) = e^x \Rightarrow f^{(k)}(0) = 1 \Rightarrow a_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{k!}$$



2/17/10 (cont.)

Qx (cont.)

$$\begin{aligned}\text{so } f(x) = e^x &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x\end{aligned}$$

□ we can also determine the radius of convergence for such a series:

$$\frac{|c_{k+1}|}{|c_k|} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{1}{k+1} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{|c_{k+1}|}{|c_k|} = \lim_{k \rightarrow \infty} \frac{1}{k+1} = 0$$

⇒ therefore $R = \infty$, meaning that the infinite series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} \text{ converges to } e^x \text{ for all values of } x$$

□ consider how useful this might be! for example, suppose someone asked you to prove that $\frac{d}{dx} e^x = e^x$:

$$\begin{aligned}\frac{d}{dx} (e^x) &= \frac{d}{dx} \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \right) \\ &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ &= e^x\end{aligned}$$

□ so we can write a general expression for the Taylor series of a function:

$$\begin{aligned}f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n\end{aligned}$$

□ note that we can estimate $e^{0.1}$ without a calculator!

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since 0.1 is close to zero, $e^x \approx 1 + x + \frac{x^2}{2} + \dots$

one term: $e^{0.1} \approx 1$

two terms: $e^{0.1} \approx 1 + 0.1 = 1.1$

three terms: $e^{0.1} \approx 1 + 0.1 + \frac{1}{2}(0.1)^2 = 1.105$

four terms: $e^{0.1} \approx 1 + 0.1 + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 = 1.10517$

→ 'higher order' terms contribute less and less (true value is 1.105170918...)

⇒ thus, by considering the first few terms in the Taylor series (the book calls this the Taylor polynomial), we have a very useful means of approximating a function!

□ we can also use our Taylor series expansion for e^x for related functions without solving for all the derivatives (which can get messy), but simply make a substitution

$$\text{we have } e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{u^k}{k!}$$

if we let $u \equiv x^2$, then this becomes

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \dots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k}}{k!} = e^{x^2}$$

(this beats computing all the derivatives of e^{x^2} !!)

action: