

12/16/16

Additional Examples of Power Series Intervals of Convergence

ex find R for $\sum_{n=0}^{\infty} (2n)! (x-1)^n$

$$\frac{|c_{n+1}|}{|c_n|} = \frac{(2(n+1))!}{(2n)!} = \frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)(2n)(2n-1)\dots(2)(1)}{(2n)(2n-1)\dots(2)(1)} = (2n+2)(2n+1)$$

$$\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n \rightarrow \infty} (2n+2)(2n+1) = \infty = k \Rightarrow R=0$$

$\leadsto$ so this series will only converge for a single value of x , namely $x=1$

$\therefore$

ex find R for $\sum_{n=1}^{\infty} \frac{(x+4)^n}{2^n \cdot n}$

$$\frac{|c_{n+1}|}{|c_n|} = \frac{\frac{1}{2^{n+1}(n+1)}}{\frac{1}{2^n \cdot n}} = \frac{n \cdot 2^n}{(n+1) \cdot 2^{n+1}} = \frac{n}{2(n+1)}$$

$$\lim_{n \rightarrow \infty} \frac{|c_{n+1}|}{|c_n|} = \lim_{n \rightarrow \infty} \frac{n}{2(n+1)} = \frac{1}{2}$$

* Aside: if you are unsure about evaluating a limit, use l'Hopital's rule:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} \quad \text{provided that } \lim_{x \rightarrow \infty} f(x), g(x) = \pm \infty \text{ or } 0$$

$\leadsto k = \frac{1}{2}$, so $R=2$; so this series converges for $|x+4| < 2$,
or on the interval $-6 < x < -2$

NOTE: we are not sure what happens at the bounds of the interval though (i.e. at $x=-6$ or $x=-2$), but we could simply plug in those values and evaluate convergence using another method [doing such allows you to make a more definitive statement that convergence is ensured for $-6 \leq x < -2$]

$\therefore$

again, to emphasize, our method for determining R only tells us what happens within the specified interval, but not at the boundaries (i.e. at $x = a \pm R$); to determine such, you need to evaluate the specific series using a different test/theorem to verify the entire interval of convergence (e.g. see last example)

ex) determine R for $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ $\left[= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right]$
(see below)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left| \frac{(-1)^{n+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \right|}{\left| \frac{(-1)^n x^{2n+1}}{(2n+1)!} \right|} = \frac{(2n+1)! |x^{2n+3}|}{(2n+3)! |x^{2n+1}|} = \frac{|x^2|}{(2n+3)(2n+2)}$$

So $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{(2n+3)(2n+2)} |x^2| = 0 \quad (\text{no matter what } x \text{ is})$
 $= k$

$\Rightarrow R = \infty$ (this series will converge for all values of x)

* Aside: this is the series expansion for $\sin(x)$, a point we will come back to soon

ex) A Bessel function of order 0 is defined as $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$

Aside: first defined by Bernoulli (1700-1782) as a means to solve the following ODE:

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha^2) y = 0$$

ex) find the domain for $J(x)$ by determining the radius of convergence for the power series

$$\frac{|a_{n+1}|}{|a_n|} = \frac{\left| \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} [(n+1)!]^2} \right|}{\left| \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2} \right|} = \frac{2^{2n} (n!)^2 |x^{2n+2}|}{2^{2n+2} [(n+1)!]^2 |x^{2n}|} = \frac{|x^2|}{2^2 (n+1)^2}$$



2/16/10

ex (cont.)

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x|^2}{4(n+1)^2}$$

$$\text{so } \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{x^2}{4(n+1)^2} = 0 = k \Rightarrow R = \frac{1}{k} = \infty$$

thus the domain of $J_0(x)$ is the entire x -interval

b) determine $J_0(0)$

$$J_0(0) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 0^{2n}}{2^{2n} (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n}}{4^n (n!)^2} = 1 - \frac{0^2}{4} + \frac{0^4}{64} - \dots = 1$$

$$\Rightarrow J_0(0) = 1 \quad (\text{since } x=0, 0^{2n} \neq 0 \text{ only for } n=0)$$

ex The series $\sum C_n x^n$ converges at $x = -5$ and diverges at $x = 7$. What can you say about R ?

~~answer~~ $\bullet$ -5 is within, so the interval of convergence must be at least $-5 \leq x < 5$

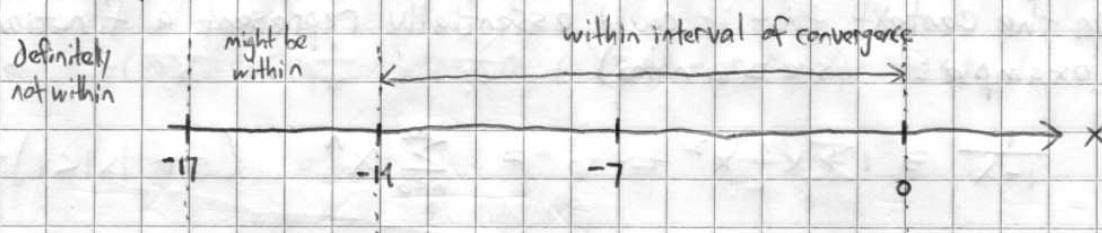
$\bullet$ we know that $x = 7$ is not, so $|x|$ might extend past 5, but not 7

$\Rightarrow$ so we must have $5 \leq R < 7$ (but since we don't know what happens beyond $|x| > 5$, we can't say $5 \leq R \leq 7$)

2/16/10

ex Similar to the last question, $\sum C_n (x+7)^n$ converges at $x=0$ and diverges at $x=-17$. What can we say about R ?

→ draw a picture!



→ $7 \leq R < 10$

ex Suppose $\sum C_n x^n$ converges at $x=-4$ and diverges when $x=7$. Then

- $\textcircled{A} x=10 \rightarrow$ diverges
- $\textcircled{B} x=3 \rightarrow$ converges
- $\textcircled{C} x=1 \rightarrow$ converges
- $\textcircled{D} x=6 \rightarrow$ not possible to conclude about convergence given the present amount of information

ex True or false: If the power series $\sum C_n x^n$ converges at $x=10$, then it must converge at $x=-10$

False - we could have the case where $R=10$ and the series converges on one endpoint ($x=10$) but not the other ($x=-10$)