

2/13/10

MATH 250B

Question : How can we determine R for a given power series?

→ sometimes it can be relatively straight-forward

ex find R for $\sum_{n=0}^{\infty} (\frac{1}{2})^n x^n$

→ geometric series w/ a scale factor of unity and a common ratio of $\frac{1}{2}$

$$\sum_{n=0}^{\infty} (\frac{1}{2})^n x^n = \sum_{n=0}^{\infty} (\frac{x}{2})^n = \begin{cases} \frac{1}{1-x/2} & \text{for } |x| < 2 \\ \infty & \text{for } |x| \geq 2 \end{cases}$$

⇒ $R = 2$ (value of x outside $|x| < 2$ will cause the series to diverge)

ex Using similar logic, we can find R for $\sum_{n=0}^{\infty} (\frac{1}{4})^n (x-3)^n$

$$\sum_{n=0}^{\infty} (\frac{1}{4})^n (x-3)^n = \sum_{n=0}^{\infty} (\frac{x-3}{4})^n = \begin{cases} \frac{1}{1-\frac{x-3}{4}} & \text{for } |\frac{x-3}{4}| < 1 \\ \infty & \text{for } |\frac{x-3}{4}| > 1 \end{cases}$$

→ so the series converges when $|x-3| < 4$, meaning that $R = 4$ when the power series is centered about $x = 3$ (i.e. convergence for $-1 < x < 7$)

General Method for Finding R

- more generally, we can find the radius of convergence for a given power series by applying the ratio test (see 2/5/10 notes)

Consider the power series $\sum_{n=0}^{\infty} C_n(x-a)^n \equiv \sum_{n=0}^{\infty} a_n$

→ applying the ratio test, the ratio of successive terms is then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|C_{n+1}(x-a)^{n+1}|}{|C_n(x-a)^n|} = \frac{|C_{n+1}|}{|C_n|} |x-a|$$

if $\lim_{n \rightarrow \infty} \frac{|C_{n+1}|}{|C_n|} = K$, then $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = K \cdot |x-a|$

so $K|x-a| < 1 \Rightarrow \text{convergence}$

$|x-a| < \frac{1}{K} \Rightarrow \text{convergence}$

$K|x-a| > 1 \Rightarrow \text{divergence}$

OR

$|x-a| > \frac{1}{K} \Rightarrow \text{divergence}$

→ for simplicity, we therefore let $R \equiv \frac{1}{K}$

- so to summarize, if $\lim_{n \rightarrow \infty} \frac{|C_{n+1}|}{|C_n|} = K$, then the radius of convergence for the power series $\sum_{n=0}^{\infty} C_n(x-a)^n = \sum_{n=0}^{\infty} a_n$

is just $R = \frac{1}{K}$ such that if $x < a-R$ or $x > a+R$, the series diverges

• NOTE: if $K = \infty$, then $R = 0$ (the series converges only at $x=a$)

• if $K = 0$, then $R = \infty$ (the series converges for all values of x)

• notice that $\frac{|C_{n+1}|}{|C_n|} \neq \frac{|a_{n+1}|}{|a_n|}$, so make sure to be clear/careful w/ your notation!!

12/15/10 (cont.)

ex] Find the interval of convergence for $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ (power series centered about the origin)

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1}{(n+1) \cdot n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1}$$
$$= \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0 = k \Rightarrow R = \infty$$

→ so this series converges for all values of x (because the $\frac{1}{n!}$ decreases faster than the exponential x^n with increasing n , no matter what the value of x is)

ex] Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$

Find R , and also determine what happens for $x=1$

$$|a_n| = \frac{x^{2n-1}}{2n-1}, \text{ so } \frac{|a_{n+1}|}{|a_n|} = \frac{\frac{x^{2(n+1)-1}}{2(n+1)-1}}{\frac{x^{2n-1}}{2n-1}} = \frac{x^{2n+1}}{2n+1} \cdot \frac{2n-1}{x^{2n-1}} = \frac{2n-1}{2n+1} x^2$$

$$\text{So } \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} x^2 = x^2$$

$$\text{or } \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{2n-1}{2n+1} = 1 = k \Rightarrow R = 1$$

→ so the series converges if $x^2 < 1$, or simply $|x| < 1$. So while this particular series is 'centered about' $x=0$, we don't know if things actually converge for $x=1$ or not. To determine such, we simply plug in $x=1$ and evaluate:

$$\text{for } x=1, \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1}$$
$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \dots$$

→ as you might infer, the ratio test is not going to help us here (we already used such to determine $R!$), which is going to be the case when on the bounding ends of the interval of convergence, so we'll need to use something else

consider the alternating series test (2/5 notes):

$$a_n = \frac{1}{2n-1}$$

$$\lim_{n \rightarrow \infty} \frac{1}{2n-1} = 0$$

~~the ratio test is not going to help us here~~

$$a_{n+1} = \frac{1}{2(n+1)-1} = \frac{1}{2n+1} < \frac{1}{2n-1} = a_n$$

→ so by the alternating series test, $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1}$ converges

→ therefore, the power series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$ converges for $x=1$ (a boundary point on the interval of convergence for this series)

$$\Rightarrow \text{recall that } \arctan(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1} \quad (2/10 \text{ notes})$$

and since we know that this series converges for $x=1$, using the series expansion for $\arctan(1)$ is valid, thereby allowing us to properly deduce

$$\pi = 4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1}$$

NOTE: since $x=1$ is on the boundary of convergence, while this series above does indeed converge to π , it does so relatively 'slowly', thus it is not a very good means to estimate π computationally (there are other series expansions for π that exist which converge much more rapidly)

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