

1250B

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Power Series + Series Expansions [ref. Leah Edelstein-Keshet notes]

- remember when we 'summed up' all the terms in the infinite geometric series, we found

$$S = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad (\text{provided } |x| < 1)$$

- so now consider the argument in the reverse direction: if we have the function

$$f(x) = \frac{1}{1-x} \quad (\text{and } |x| < 1) \rightarrow f(x) = 1 + x + x^2 + \dots$$

$\Rightarrow$ that is, we can express the function $f(x) = \frac{1}{1-x}$ as an infinite series (of successively higher order polynomials)

- note that this conclusion will be true even if $x < 0$ (provided $|x| < 1$):

$$\begin{aligned} \frac{1}{1-(-x)} &= \frac{1}{1+x} = 1 + (-x) + (-x)^2 + (-x)^3 + (-x)^4 + \dots \\ &= 1 - x + x^2 - x^3 + x^4 - \dots \end{aligned}$$

$$\begin{aligned} \leadsto \text{leading us to conclude that } \frac{1}{1+x} &= 1 - x + x^2 - x^3 + x^4 - x^5 + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

- so given the equality, there are all sorts of neat things we can do now, for example:

$$\begin{aligned} \int_0^x \frac{1}{1+t} dt &= \ln|1+t| \Big|_0^x = \ln|1+x| \\ &= \int_0^x (1 - t + t^2 - t^3 + t^4 - \dots) dt = \left(t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \dots \right) \Big|_0^x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots \end{aligned}$$

we simply substituted in an series representation for the function



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otherwise the series won't converge

thereby allowing us to conclude (provided still that $|x| < 1$):

$$\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

□ another example, since $\frac{1}{1-x} = 1+x+x^2+x^3+\dots$, we 'replace' x with $-t^2$ (still, assuming $|-t^2| < 1$):

$$x = -t^2 \rightarrow \frac{1}{1-(-t^2)} = \frac{1}{1+t^2} = 1 + (-t^2) + (-t^2)^2 + (-t^2)^3 + \dots$$

$$= 1 - t^2 + t^4 - t^6 + t^8 - \dots$$

and similarly we can integrate:

$$= \sum_{n=0}^{\infty} (-1)^n t^{2n}$$

$$\int_0^x \frac{1}{1+t^2} dt = \arctan(t) \Big|_0^x = \arctan(x)$$

$$= \int_0^x [1 - t^2 + t^4 - t^6 + t^8 - \dots] dt$$

$$= \left[t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \dots \right] \Big|_0^x$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$$

$\Rightarrow$ so provided $|x| < 1$ (thus $|x^2| < 1$),

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{2n-1}$$

ex (a practical extension)

$$\arctan(1) = \frac{\pi}{4} \quad (\text{by definition})$$

$$(\text{by series expansion}) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\text{thus } \pi = 4 \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) \Rightarrow$$

remember that we required $|x| < 1$ for the series to be true, but here we have $x=1$; we will come back later to this apparent discrepancy...

See 1/27/10 notes re. the Gregory-Leibniz series

$$\pi = 4 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n-1}$$

so we were able to come up w/ a closed form expression for the irrational # π !

□ let's take a step back for a moment and reconsider

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \text{ provided } |x| < 1$$

→ we call this representation of the function $f(x) = \frac{1}{1+x}$ the **TAYLOR SERIES** expansion of $f(x)$, essentially representing the function as an infinite sum of polynomials

→ the apparent 'choice' of polynomials indicates a given choice for a set of **BASIS FUNCTIONS**

→ the fact that we needed $|x| < R$ (where $R=1$ in our case) indicates that R is our **RADIUS OF CONVERGENCE** (outside of $|x| > R$, the infinite series diverges)

→ a Taylor series is a special case of a more general type of series called a **POWER SERIES**

DEFINITION

A power series is defined as having the form:

$$C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \dots + C_n(x-a)^n + \dots$$

$$= \sum_{n=0}^{\infty} C_n(x-a)^n$$

where C_n is some sequence dependent upon n , a is a constant and x is a variable quantity (not dependent upon n)

→ for a fixed value of x , if $\lim_{n \rightarrow \infty} S_n = L$ (where S_n is the sequence of partial sums, i.e. the sum of the first n terms of the power series), we say that the power series **converges** to L for that value of x

POWER SERIES

$$\bullet x - x^3 + x^6 - x^{10} + x^{15} - \dots$$

$$\bullet x^7 + x + 2$$

NOT POWER SERIES

$$\bullet \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \dots \quad \left(\text{negative powers of } x \right)$$

$$\bullet 1 + x + (x-1)^2 + (x-2)^2 + (x-3)^2 + \dots \quad \left(\text{each term a power to a diff. quantity} \right)$$