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Theorem 9.7 (Ratio Test) [Everyone's favorite for some reason]

Idea: the ratio between successive terms can tell you whether higher order terms get small quickly enough such that their contribution will be negligible

For the series $\sum a_n$, suppose the ratio of sequences $\frac{|a_{n+1}|}{|a_n|}$ has a limit

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = L$$

- if $L < 1$, then $\sum a_n$ converges
- if $L > 1$ or is infinite, $\sum a_n$ diverges
- if $L = 1$, the ratio test doesn't tell us anything about the convergence
 ↗ this happens often, so be careful...

☐ determine whether $\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)/4^{n+1}}{n/4^n} = \frac{n+1}{4n}, \text{ so } \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{n+1}{4n} = \frac{1}{4}$$

$\Rightarrow$ ratio test indicates $\sum_{n=1}^{\infty} \frac{n}{4^n}$ converges $\therefore$

$$\text{☐ } \sum_{n=1}^{\infty} \frac{(-4)^n}{n!} \rightsquigarrow \frac{|a_{n+1}|}{|a_n|} = \left| \frac{(-4)^{n+1}/(n+1)!}{(-4)^n/n!} \right| = \frac{n!}{(n+1)!} \left| \frac{(-4)^{n+1}}{(-4)^n} \right|$$

$$= 4 \frac{n!}{(n+1)!} = \frac{4}{n+1} \quad [\text{NOTE: } \frac{n!}{(n+1)!} = \frac{1 \cdot 2 \cdot 3 \cdots n}{1 \cdot 2 \cdot 3 \cdots n \cdot (n+1)} = \frac{1}{n+1}]$$

$$\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{4}{n+1} = 0 \Rightarrow \text{ratio test implies } \sum_{n=1}^{\infty} \frac{(-4)^n}{n!} \text{ converges } \therefore$$

$$\boxed{\text{ex}} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} \quad \rightsquigarrow \quad \frac{|a_{n+1}|}{|a_n|} = \left| \frac{(-1)^{n+1}/\sqrt[3]{n+1}}{(-1)^n/\sqrt[3]{n}} \right| = \frac{n^{1/3}}{(n+1)^{1/3}} = \sqrt[3]{\frac{n}{n+1}}$$

$\lim_{n \rightarrow \infty} \sqrt[3]{\frac{n}{n+1}} = \sqrt[3]{1} = 1 \quad \Rightarrow$ ratio test doesn't tell us anything about the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$
(we need to use something else) $\therefore$

NOTE: in some of these examples, the sign of successive terms kept flipping back + forth; we call these **alternating series**

Theorem 9.6

If $\sum |a_n|$ converges, then so does $\sum a_n$
(we say in this case that $\sum a_n$ converges absolutely)

$$\boxed{\text{ex}} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} \quad |a_n| = \frac{1}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^3} \rightarrow \text{converges} \quad (p\text{-series w/ } p=3)$$

$\rightsquigarrow$ so by Theorem 9.6, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3}$ converges (absolutely) $\therefore$

DEFINITION

□ $\sum a_n$ is absolutely convergent if both $\sum a_n$ and $\sum |a_n|$ converge

□ $\sum a_n$ is conditionally convergent if $\sum a_n$ converge but $\sum |a_n|$ diverges

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}, \quad \sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$

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Theorem 9.8 (Alternating Series Test)

Idea: Sometimes if successive terms alternate sign, they can ultimately end up canceling one another out

A series of the form $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$

converges if $0 < a_{n+1} < a_n$ for all n AND $\lim_{n \rightarrow \infty} a_n = 0$

ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ • first try applying Theorem 9.6: $|a_n| = \left| \frac{(-1)^{n-1}}{n} \right| = \frac{1}{n}$
and $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges $\Rightarrow$ Th. 9.6 doesn't apply here

• here, $a_n = \frac{1}{n}$, so $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$
 $a_{n+1} = \frac{1}{n+1} < \frac{1}{n} = a_n$ } NOTE: both these conditions must be satisfied for Th. 9.8 to apply

$\rightarrow$ since both conditions are satisfied, the Alternating series test implies that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges (but note that it is conditionally convergent!)

ex $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n} \rightsquigarrow a_n = \frac{1}{e^n}$ $\lim_{n \rightarrow \infty} \frac{1}{e^n} = 0 \checkmark$

$a_{n+1} = \frac{1}{e^{n+1}} < \frac{1}{e^n} = a_n \checkmark$

$\Rightarrow$ both conditions satisfied, so Th. 9.8 implies $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{e^n}$ converges

ex $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ Rewrite as: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} (-1) \frac{(-1)^{n+1}}{\sqrt[3]{n}} = - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n}}$

$a_n = n^{-1/3} \rightarrow \lim_{n \rightarrow \infty} n^{-1/3} = 0 \checkmark$

$a_{n+1} = \frac{1}{\sqrt[3]{n+1}} < \frac{1}{\sqrt[3]{n}} = a_n$

$\Rightarrow$ Th. 9.8 indicates $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$ converges