

□ as we have done up to this point, we can compare an infinite series to an indefinite integral to help determine convergence

Theorem 9.3 (Integral Test)

Suppose $a_n = f(n)$ and $f(x)$ is decreasing for $x \geq c$

- if $\int_c^\infty f(x) dx$ converges, then $\sum_{n=c}^\infty a_n$ converges
- if $\int_c^\infty f(x) dx$ diverges, then $\sum_{n=c}^\infty a_n$ diverges

(see examples 2 and 3 in H.H. pg. 452-453)

ex does the series $\sum_{n=1}^\infty \frac{1}{e^n}$ converge?

$\leadsto \frac{1}{e^n}$ suggest $f(x) = xe^{-x}$, which on the interval $[1, \infty)$ is continuous and decreasing

$$\int_1^\infty xe^{-x} dx = \lim_{b \rightarrow \infty} \int_1^b xe^{-x} dx \quad \text{[use integration by parts: } u=x \rightarrow u'=1, v=e^{-x} \rightarrow v'=-e^{-x}]$$

$$= \lim_{b \rightarrow \infty} \left[-xe^{-x} \Big|_1^b - \int_1^b -e^{-x} dx \right] = \lim_{b \rightarrow \infty} \left[\frac{0}{e} - \frac{b+1}{e^b} \right] = \frac{0}{e}$$

so $\int_1^\infty xe^{-x} dx = \frac{0}{e}$, thus by the integral test $\sum_{n=1}^\infty \frac{1}{e^n}$ converges

NOTE: we however can not conclude that $\sum_{n=1}^\infty \frac{1}{e^n} = \frac{0}{e}$

[not only does theorem 9.3 not say such, in all but trivial cases the LEFT or RIGHT Riemann sum never equals the integral here because we are essentially using a fixed interval of one] $\therefore$

2/3/10 (cont.)

Theorem 9.4 (Comparison Test)

Idea: compare to a series whose behavior we already know

Suppose $0 \leq a_n \leq b_n$ (for any given value of n)

- if $\sum b_n$ converges, then $\sum a_n$ converges
- if $\sum a_n$ diverges, then $\sum b_n$ diverges

☐ $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$

$\leadsto$ when n gets big, $\frac{1}{n^2+3n} \approx \frac{1}{n^2}$, so we might expect convergence and thus look for a 'function' bigger than that we know converges

• $n^2+3n \geq n^2$ for $n \geq 1$, so $\frac{1}{n^2+3n} < \frac{1}{n^2}$

• since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series w/ $p > 1$), the comparison test says that $\sum_{n=1}^{\infty} \frac{1}{n^2+3n}$ converges

[but note that we don't presently know what it converges to!!]

☐ check/determine convergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$

• for large n , $\frac{1}{\sqrt{n+4}} \approx \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ leading us to suspect divergence

• so we need $a_n \geq b_n$ where $a_n = \frac{1}{\sqrt{n+4}}$

• for $n \geq 1$, $n > n+4$, so $\frac{1}{\sqrt{n+4}} \geq \frac{1}{3\sqrt{n}}$

• $\sum_{n=1}^{\infty} \frac{1}{3\sqrt{n}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ (divergent p-series) $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}}$ is

divergent by comp. test $\therefore$

Theorem 9.5 (Limit Comparison Test)

Idea: avoids the inequalities of the comparison test

Suppose $a_n > 0$ and $b_n > 0$ for all n

if $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ where $c > 0$

then the two series $\sum a_n$ and $\sum b_n$ either both converge or diverge

ex determine whether $\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+4}$ converges or not

• for large n , $\frac{\sqrt[3]{n}}{n+4} \approx \frac{n^{1/3}}{n} = \frac{1}{n^{2/3}}$ (likely diverges)

• we have $a_n = \frac{\sqrt[3]{n}}{n+4}$ and let $b_n = \frac{1}{n^{2/3}}$

• then $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n}{n+4} = 1$

• since $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (p -series w/ $p \leq 1$), then

$\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{n+4}$ diverges by the limit comp. test $\therefore$

ex $\sum_{n=1}^{\infty} \frac{1}{n+2^n}$ [be careful to stop + think what 'test' might be best]

• for large n , $\frac{1}{n+2^n} \approx \frac{1}{2^n}$ (will likely converge since seq. goes to zero quickly)

• so for $n \geq 1$, $\frac{1}{n+2^n} \leq \frac{1}{2^n}$, noting that $\frac{1}{2^n} = \left(\frac{1}{2}\right)^n$

• $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = 2$ (convergent geometric series)

• since $\frac{1}{n+2^n} \leq \frac{1}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges, the comparison test

implies that $\sum_{n=1}^{\infty} \frac{1}{n+2^n}$ converges $\therefore$