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MATH 250B

consider relating to waves
(or 'modes') on a vibrating
string

QUESTION: does the following series converge?

$$S = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

(this is called the **HARMONIC** series)

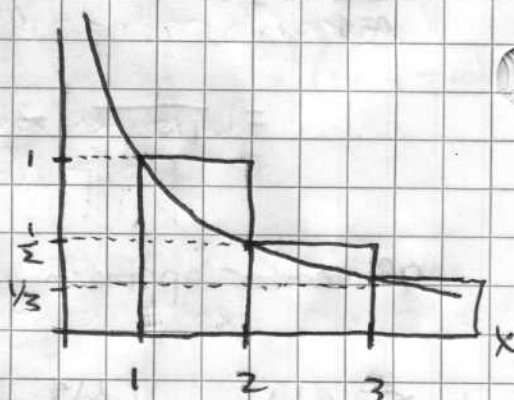
→ seems reasonable that it might converge, since successive terms are perpetually getting smaller (and thus eventually negligible)

□ however, consider the integral $\int_1^{\infty} \frac{1}{x} dx$, which we know diverges; consider using the **LEFT** rule for a Riemann sum w/ a step-size of 1

→ since $\frac{1}{x}$ is decreasing w/ x , then

$$\int_1^{\infty} \frac{1}{x} dx < \sum_{n=1}^{\infty} \text{LEFT} = \sum_{n=1}^{\infty} \frac{1}{n}$$

(can see this clearly in the figure)



⇒ therefore $\sum_{n=1}^{\infty} \frac{1}{n}$ will diverge

[pause to consider that there is a deep connection among all the things we have learned up to this point!]

□ so along these lines, we will look @ the general notion of whether an infinite series converges or not (i.e. a series need not be 'harmonic' or 'geometric', but we can look at more general cases)

DEFINITION: Assume $S_N \equiv \sum_{n=0}^{N-1} a_n$. If $\lim_{N \rightarrow \infty} S_N = S$ (i.e. some # that is not infinite), then we say the series converges. If S does not exist (i.e. $S = \pm \infty$), then we say the series diverges.

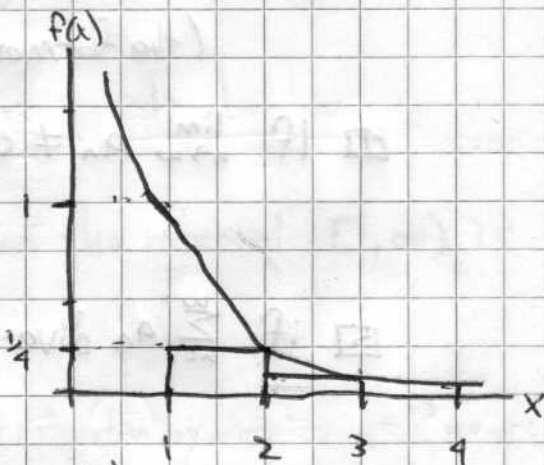
ex) does the series $\sum_{n=0}^{\infty} (1 - e^{-n})$ converge?

$\lim_{n \rightarrow \infty} (1 - e^{-n}) = 1 \neq 0$, so we will ultimately (for large n) just keep adding up ones, indicating things must diverge $\therefore$

ex) does the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \dots$ converge?

• rewrite as $S = \sum_{n=1}^{\infty} \frac{1}{n^2}$

• relate back to the integral $\int_1^{\infty} \frac{1}{x^2} dx$ using the RIGHT Riemann sum w/ a step-size of one



$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$\text{but } \sum_{n=2}^{\infty} \frac{1}{n^2} < \int_1^{\infty} \frac{1}{x^2} dx \text{ (which converges)}$$

$$\leadsto \sum_{n=2}^{\infty} \frac{1}{n^2} \text{ must converge } \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges } \therefore$$

p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \iff \int_1^{\infty} \frac{1}{x^p} dx$$

Rule: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

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Series Convergence Properties

note similarity to rules for improper integrals + convergence!!

1 if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

2 if k is a constant, then $\sum_{n=1}^{\infty} k a_n$ converges to $k \sum_{n=1}^{\infty} a_n$

3 changing a finite # of terms does not (necessarily) change whether the series converges or not, but it may change the limit if it does converge

eg. $\sum_{n=1}^{\infty} \frac{1}{n^2} \leftrightarrow \sum_{n=3}^{\infty} \frac{1}{n^2}$

Caution: be mindful of instances such as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ vs. $\sum_{n=0}^{\infty} \frac{1}{n^2}$

(the former converges while the latter diverges)

4 If $\lim_{n \rightarrow \infty} a_n \neq 0$ or the limit does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges

5 if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} k a_n$ diverges (provided $k \neq 0$)

ex does $\sum_{n=0}^{\infty} \frac{3}{n+2}$ converge?

$$\sum_{n=0}^{\infty} \frac{3}{n+2} = 3 \sum_{n=0}^{\infty} \frac{1}{n+2} \quad (\text{let } m = n+2) = 3 \sum_{m=2}^{\infty} \frac{1}{m}$$

$$= 3 \left[\sum_{m=1}^{\infty} \frac{1}{m} - 1 \right] \rightsquigarrow \text{which is a divergent } p\text{-series}$$

$\Rightarrow$ diverges

$\therefore$