

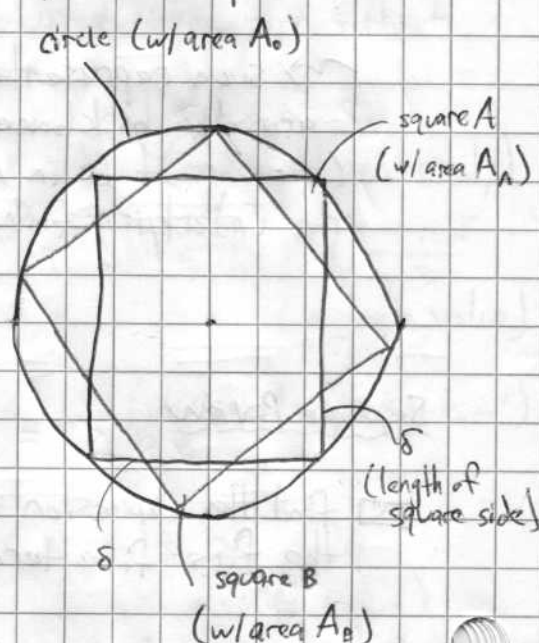
1/29/10 (MM)

→ think back to Zeno's paradox w/ the race between Achilles and the tortoise and you have a situation analogous to a geometric series (where the common ratio is $\sim \frac{1}{2}$, but not quite since the tortoise keeps moving ahead!)

□ think about using squares to estimate the area of a circle:

first approx: $A_0 \approx A_A$ (not so good!)

second approx: $A_0 \approx A_A + A_B$ (excluding overlap w/ square A)
 $= A_A + A_{B'}$



→ where you could envision using geometry to come up w/ an algebraic expression for $A_{B'}$

⇒ the second approx. would clearly be better than the first, and you could envision adding a third square to do even better and so on, where eventually in the limit of using an infinite # of squares, you would get exactly the area of a circle!

□ note that we can express the geometric series in terms of a single 'geometric sequence' term:

~~XXXXXXXXXX~~ $a_n = a x^{n-1}$ (or recursively as $a_n = x a_{n-1}$)

→ then the geometric series will be just $S_N = \sum_{n=0}^{N-1} a_n$

NOTE: $S_N = \sum_{n=0}^{N-1} a_n \iff S_N = \sum_{n=1}^N a_n \rightarrow S_N$ is the sum of N terms, but you can sometimes offset that starting index as needed

1/29/10

- in the case of the geometric series, we can actually write down a simple closed form expression for S_N (even in the infinite case!); or put another way, we don't need to add all the terms together

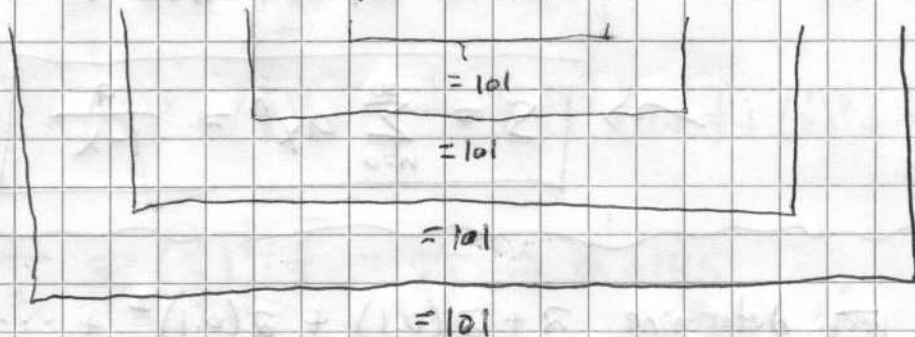
HISTORICAL ASIDE

Carl Friedrich Gauss (1777-1855, German mathematician/physicist) was in grade school one day when his teacher, supposedly wanting a chunk of time to himself, told the students to keep themselves busy by computing the sum of the first 100 integers (let's call it M); remarkably just a few moments after posing the problem, young Gauss raised his hand w/ the correct answer (much to the consternation of his teacher!)

→ how did he do it? obviously he didn't compute the sum directly

- Gauss noticed an underlying symmetry to the problem

$$M = \sum_{n=1}^{100} n = 1 + 2 + 3 + 4 + \dots + 97 + 98 + 99 + 100$$



$$= 101 \cdot 50 \text{ (since there will be 50 such pairs)} = 5050$$

OR (more generally)

where $a, b \in \mathbb{N}$ and $b > a > 0$

$$\text{let } N = \sum_{n=a}^b n = a + (a+1) + (a+2) + \dots + (b-2) + (b-1) + b$$

$$\text{then } N = \frac{1}{2} [b^2 - a^2 + a + b] = \frac{N}{2} (a+b)$$

(where $N = b - a + 1$)

11/29/10

back to the geometric series, we can use analogous reasoning to find a general / closed form expression for the sum.

$$S_N = a + ax + ax^2 + \dots + ax^{N-1} \quad (\text{multiply both sides by } x)$$

$$xS_N = ax + ax^2 + ax^3 + \dots + ax^N \quad (\text{now subtract } S_N)$$

$$\begin{aligned} xS_N - S_N &= (ax + ax^2 + \dots + ax^N) - (a + ax + \dots + ax^{N-1}) \\ &= ax^N - a = S_N(x-1) \end{aligned}$$

$$\rightarrow \boxed{S_N = \frac{ax^N - a}{x-1} = a \frac{(1-x^N)}{1-x} = \sum_{n=0}^{N-1} ax^n}$$

$\Rightarrow$ not too hard to see what happens in the infinite case (i.e. $N \rightarrow \infty$) either:

$$\text{if } |x| < 1, \lim_{N \rightarrow \infty} x^N \rightarrow 0$$

$$\text{but if } |x| > 1, \lim_{N \rightarrow \infty} x^N \rightarrow \infty$$

$$\rightarrow \boxed{S = \sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}}$$

(provided $|x| < 1$, otherwise the series will diverge)

ex determine $2 + 2(0.1) + 2(0.1)^2 + \dots + 2(0.1)^{10}$

$\rightarrow$ 11 terms, which we can write as $S_{11} = \sum_{n=0}^{10} 2(0.1)^n$

$$\Rightarrow S_{11} = 2 \frac{(1-0.1^{11})}{1-0.1} \approx 2.22$$

$\therefore$

$$\boxed{\text{ex}} \quad 3 + \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \dots + \frac{3}{2^{10}}$$

$$= 3 + 3\left(\frac{1}{2}\right) + 3\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \dots + 3\left(\frac{1}{2}\right)^{10}$$

$$= \sum_{n=0}^{10} 3\left(\frac{1}{2}\right)^n = 3 \frac{(1-0.5^{11})}{1-0.5} \approx 5.997 \quad \therefore$$

$$\boxed{\text{ex}} \quad \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$$

→ in this case we have $a=1$ and $x=\frac{1}{3}$, but note that we don't start @ $n=0$ and therefore need to subtract off those first few terms

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n - \sum_{n=0}^3 \left(\frac{1}{3}\right)^n$$

$$= \frac{1}{1-\frac{1}{3}} - \frac{[1-(\frac{1}{3})^4]}{1-\frac{1}{3}} \approx \frac{3}{2} - \frac{3(1-0.0123456)}{2} \approx 0.0185$$

OR another approach:

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \dots = \left(\frac{1}{3}\right)^1 \left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \dots\right]$$

$$= \left(\frac{1}{3}\right)^1 \cdot \frac{1}{1-\frac{1}{3}} = \left(\frac{1}{3}\right)^1 \cdot \frac{3}{2} = \frac{1}{2} \approx 0.0185 \quad \therefore$$