

1/25/10

Math 250B

Comparisons

Consider $\int_{-\infty}^{\infty} e^{-x^2} dx$ $\rightarrow$ does it converge or diverge?

- we don't know an anti-derivative for e^{-x^2} (the 'bell-shaped' curve), so we need a diff. approach

Δ we know that $\int_0^{\infty} e^{-x} dx = 1$
(see 1/29/10 note)

Δ for $x > 1$, $e^{-x^2} < e^{-x}$

Δ $\int_1^{\infty} e^{-x^2} dx$ is a finite #

Δ since $\int_0^1 e^{-x^2} dx = \int_1^{\infty} e^{-x^2} dx$ (due to symmetry)

and

$\int_1^{\infty} e^{-x^2} dx < \int_1^{\infty} e^{-x} dx$ and $\int_1^{\infty} e^{-x} dx$ converges.

$\Rightarrow$ then we must have that $\int_{-\infty}^{\infty} e^{-x^2} dx$ is a finite #
and therefore converges as well

□ this is an example of using a comparison method to determine convergence

ASIDE $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (there are a # of ways of showing this, the easiest to use your Matlab code and show numerically via Riemann sums)

$\rightarrow$ given the importance of this integral (e.g. a normal probability density function), it is worthwhile to ask the more general question:

$$\int_{-\infty}^{\infty} e^{-(x-a)^2/b} dx = ?$$

- where a is the new 'mean value' and b is related to what is called the 'standard deviation'
- think about how the function $e^{-(x-a)^2/b}$ graphically depends upon a and b (e.g. small b value changes the graph how?)
- thus, you can get some insight from the get-go on how $\int_{-\infty}^{\infty} e^{-x^2} dx$ might or might not depend upon a and b

[a has no effect, but we would reason that the integral should vary in some proportion to b since smaller b means less area under the curve]

- ◻ might be helpful to remember the substitutions we did back in H.H. 2.1

ex suppose $\int_0^1 f(t) dt = 3$, then find ◻ $\int_0^{0.5} f(2t) dt$ and ◻ $\int_0^1 f(1-t) dt$

◻ let $u = 2t$, then $du = 2 dt \rightarrow dt = \frac{1}{2} du$

$$\int_0^{0.5} f(2t) dt = \int_0^1 f(u) \cdot \frac{1}{2} du = \frac{1}{2} \int_0^1 f(u) du = \frac{3}{2}$$

$$\rightarrow \int_0^{0.5} f(2t) dt = \frac{3}{2}$$

◻ let $w = 1-t$, then $dw = -dt$

$$\int_0^1 f(1-t) dt = \int_1^0 f(w) \cdot -dw = -\int_1^0 f(w) dw$$

$$= \int_0^1 f(w) dw = 3 \rightarrow \int_0^1 f(1-t) dt = 3$$

[question to think about: why is that $\int_a^b f(x) dx = -\int_b^a f(x) dx$?
→ integral is not just an area, but it is a signed area]

→ back to our problem: let $t = \frac{x-a}{\sqrt{b}}$, then $dt = \frac{1}{\sqrt{b}} dx$
→ $dx = \sqrt{b} dt$

$$\int_{-\infty}^{\infty} e^{-(x-a)^2/b} dx = \int_{-\infty}^{\infty} e^{-t^2} \cdot \sqrt{b} dx = \sqrt{b} \int_{-\infty}^{\infty} e^{-t^2} dt = \boxed{\sqrt{b\pi}}$$

limits
don't change!

∴

1/25/10 (cont.)

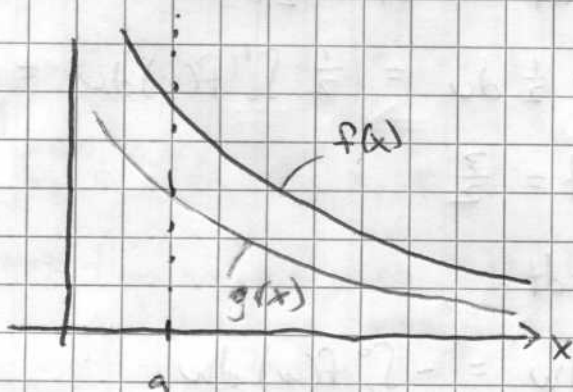
Comparison Test (HH 7.8)

→ suppose we don't know how to solve a given integral, we can try to use a different indefinite integral (whose behavior we do know) to gauge whether things diverge or not

Caveats

- making a 'comparison' can tell you whether things converge or not, but if they do, you don't typically know what they converge to
- you typically need to make an initial guess as to whether the given integral converges or not (and thus what to 'compare' to)

Basic Idea



Case 1 $\int_a^\infty g(x) dx$ convg. unknown

if $0 \leq g(x) \leq f(x)$ on $[a, \infty)$

and $\int_a^\infty f(x) dx$ converges

then $\int_a^\infty g(x) dx$ converges

Case 2 $\int_a^\infty f(x) dx$ convergence unknown

if $0 \leq g(x) \leq f(x)$ on $[a, \infty)$ and $\int_a^\infty g(x) dx$ diverges,

then $\int_a^\infty f(x) dx$ diverges

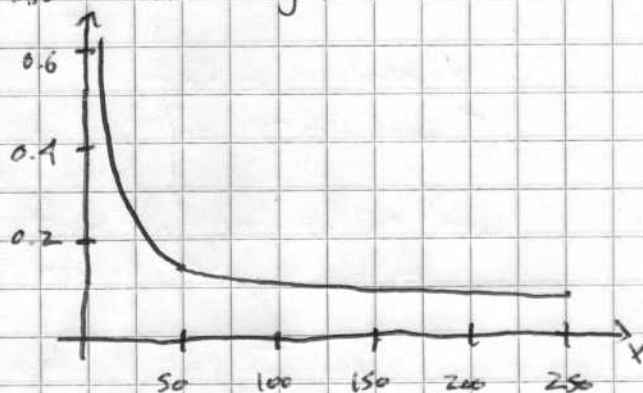
⇒ the hard part becomes determining whether or not you think the integral converges or diverges and thus whether to find a comp. function less than or greater than (respectively) with known convergence behavior

Ex $\int_1^{\infty} \frac{1}{\sqrt{x+\sqrt{x}}} dx \rightarrow$ very hard to solve outright!!

- plotting, $\frac{1}{\sqrt{x+\sqrt{x}}}$ is cont. on $[1, \infty)$

- note that $x+\sqrt{x} \leq x+x = 2x$
for $x \geq 1$

- therefore $\frac{1}{\sqrt{x+\sqrt{x}}} \geq \frac{1}{\sqrt{2x}}$ for $x \geq 1$



$$\rightarrow \int_1^{\infty} \frac{1}{\sqrt{x+\sqrt{x}}} dx \geq \int_1^{\infty} \frac{1}{\sqrt{2x}} dx = \frac{1}{\sqrt{2}} \int_1^{\infty} \frac{1}{\sqrt{x}} dx = \infty$$

$$\Rightarrow \int_1^{\infty} \frac{1}{\sqrt{x+\sqrt{x}}} dx \text{ diverges} \quad \therefore$$