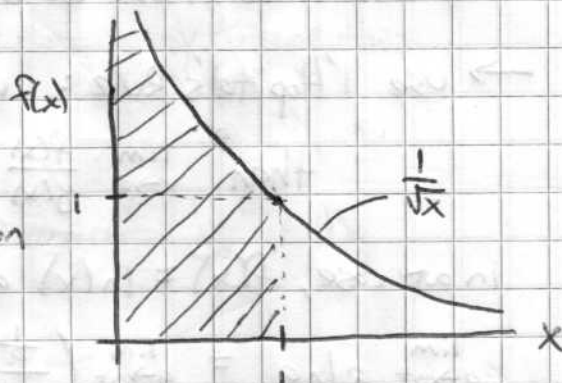


□ so what happens if the integrand blows up on the interval of integration?

ex $\int_0^1 \frac{1}{\sqrt{x}} dx$

→ same basic approach; make a substitution for the bound where things diverge and take the limit



$\lim_{x \rightarrow 0^+}$ - means to consider positive x-values getting smaller towards 0

$\lim_{x \rightarrow 0^-}$ - " negative x-values getting larger towards 0

$$\rightarrow \int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-\frac{1}{2}} dx = \lim_{a \rightarrow 0^+} [2x^{\frac{1}{2}} \Big|_a^1]$$

$$= \lim_{a \rightarrow 0^+} [2 - 2\sqrt{a}] = 2$$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{x}} dx$ converges and has a value of 2

ex $\int_0^1 \ln x dx$

□ ask a few questions:

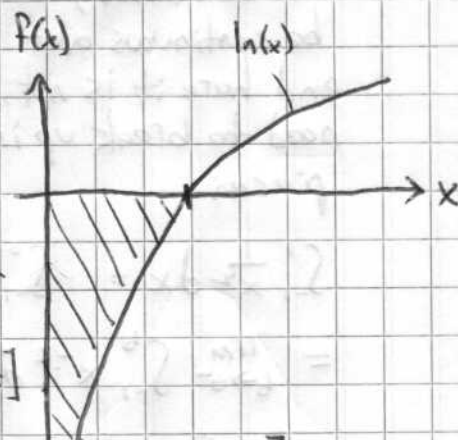
- is it improper?
 - what happens in the limit?
- } → draw a picture!

□ as $x \rightarrow 0^+$, $\ln \rightarrow -\infty$, so it is indeed improper

$$\int_0^1 \ln x dx = \lim_{a \rightarrow 0^+} \int_a^1 \ln x dx \quad [\text{integrate by parts}]$$

$$= \lim_{a \rightarrow 0^+} [x \ln x - x \Big|_a^1] = \lim_{a \rightarrow 0^+} [1 - 1 - a \ln a + a]$$

$$= \lim_{a \rightarrow 0^+} [-1 - a \ln a] \quad (?)$$



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ex (cont.) what happens to $a \ln a$ as $a \rightarrow 0^+$?

→ superficially seems analogous to asking $0 \cdot \infty$ (which apparently is undefined), but we need to be more careful...

→ use l'Hopital's Rule: when $\lim_{x \rightarrow a} f(x) = \pm \infty$ and $\lim_{x \rightarrow a} g(x) = \pm \infty$
then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$

in our case, $f(x) = \ln(x)$ and $g(x) = \frac{1}{x}$ (so the rule applies):

$$\lim_{a \rightarrow 0^+} a \ln a = \lim_{a \rightarrow 0^+} \left(\frac{\ln a}{\frac{1}{a}} \right) = \lim_{a \rightarrow 0^+} (-a) = 0$$

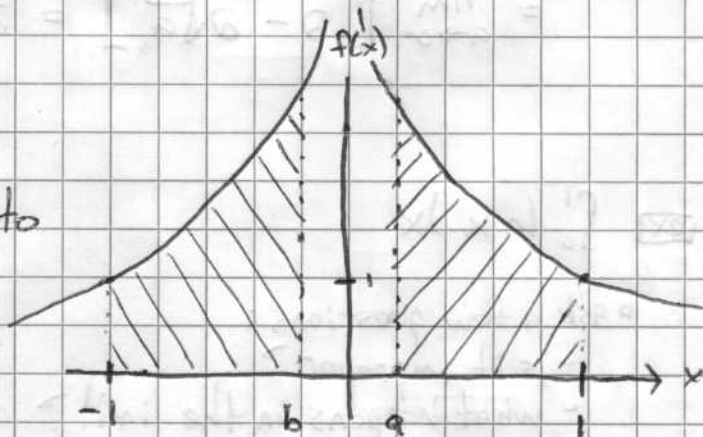
→ or put another way, as $x \rightarrow 0$, x goes to zero faster than $\ln x$ goes to $-\infty$

$$\text{so } \int_0^1 \ln x \, dx = \lim_{a \rightarrow 0^+} (-1 - a \ln a) = -1$$

so the integral converges $\therefore$

ex $\int_{-1}^1 \frac{1}{x^2} \, dx$

→ be careful; the integrand needs to be continuous over the interval and here it is not, therefore we need to break up into appropriate pieces



$$\int_{-1}^1 \frac{1}{x^2} \, dx = \int_{-1}^0 \frac{1}{x^2} \, dx + \int_0^1 \frac{1}{x^2} \, dx$$

$$= \lim_{b \rightarrow 0^-} \int_{-1}^b x^{-2} \, dx + \lim_{a \rightarrow 0^+} \int_a^1 x^{-2} \, dx$$

$$= \lim_{b \rightarrow 0^-} \left[-\frac{1}{x} \right]_{-1}^b + \lim_{a \rightarrow 0^+} \left[-\frac{1}{x} \right]_a^1 = \lim_{b \rightarrow 0^-} \left(-\frac{1}{b} - 1 \right) + \lim_{a \rightarrow 0^+} \left(-1 + \frac{1}{a} \right)$$

$$= \infty \quad \left(\text{since } \lim_{b \rightarrow 0^-} \frac{1}{b} = \lim_{a \rightarrow 0^+} \frac{1}{a} = \infty \right) \Rightarrow \text{integral diverges}$$

NOTE: common mistake (of those not careful):

$$\int_{-1}^1 \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_{-1}^1 = -2$$

→ numerically this can cause problems too (try putting this problem into your calculator)

□ another general rule that is useful: the integral $\int_0^1 \frac{1}{x^p} dx$

- converges for $p < 1$

- diverges for $p \geq 1$

⇒ you should be able to sketch out a proof for this!

$$\int_0^1 \frac{1}{x^p} dx = \lim_{a \rightarrow 0^+} \int_a^1 x^{-p} dx = \lim_{a \rightarrow 0^+} \left[\frac{x^{-p+1}}{1-p} \right] \Big|_a^1 \quad (\text{for } p \neq 1)$$

$$= \frac{1}{1-p} \lim_{a \rightarrow 0^+} \left[1 - \frac{1}{a^{p-1}} \right]$$

↓
what about $p=1$
case?