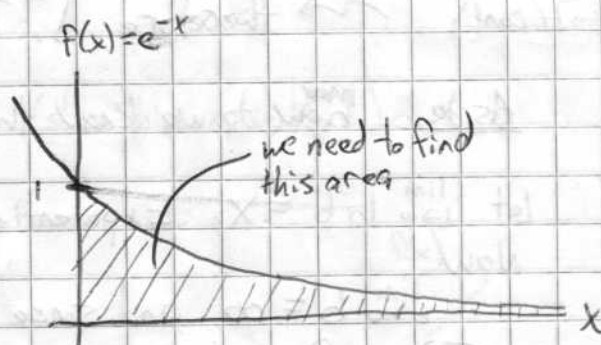


Improper Integrals (cont.)ex) consider $\int_0^{\infty} e^{-x} dx$

$$\begin{aligned}\int_0^{\infty} e^{-x} dx &= \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx \\ &= \lim_{b \rightarrow \infty} -e^{-x} \Big|_0^b = \lim_{b \rightarrow \infty} [e^0 - e^{-b}] \\ &= \lim_{b \rightarrow \infty} [1 - e^{-b}]\end{aligned}$$

Since $e^{-b} = \frac{1}{e^b} \approx 0$, then $\lim_{b \rightarrow \infty} (1 - e^{-b}) = 1$
(for large b)

$\Rightarrow$ therefore $\int_0^{\infty} e^{-x} dx$ converges and has a value of 1 $\therefore$

ex) consider $\int_1^{\infty} \frac{1}{x^2} dx$

$\rightarrow$ similar to the previous example:

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x^2} dx = \lim_{b \rightarrow \infty} -x^{-1} \Big|_1^b \\ &= \lim_{b \rightarrow \infty} \left[1 - \frac{1}{b} \right] = 1\end{aligned}$$

• so $\int_1^{\infty} \frac{1}{x^2} dx$ converges and has a value of 1

• note that interestingly, $\int_1^{\infty} e^{-x} dx = \int_1^{\infty} \frac{1}{x^2} dx$ (!)

ex) consider $\int_1^{\infty} \frac{1}{x} dx$

$$\begin{aligned}\int_1^{\infty} \frac{1}{x} dx &= \lim_{b \rightarrow \infty} \int_1^b \frac{1}{x} dx = \lim_{b \rightarrow \infty} \ln|x| \Big|_1^b \\ &= \lim_{b \rightarrow \infty} [\ln|b| - \ln 1] = \lim_{b \rightarrow \infty} \ln b = \infty\end{aligned}$$

(cont.)

1/20/10 (cont.)

□ (cont.) $\rightarrow$ therefore $\int_1^{\infty} \frac{1}{x} dx$ diverges

Aside: ^{now} how do we know that $\lim_{b \rightarrow \infty} \ln b = \infty$?

let $\lim_{b \rightarrow \infty} \ln b = X$. Exponentiate both sides, such that $b = e^X$.

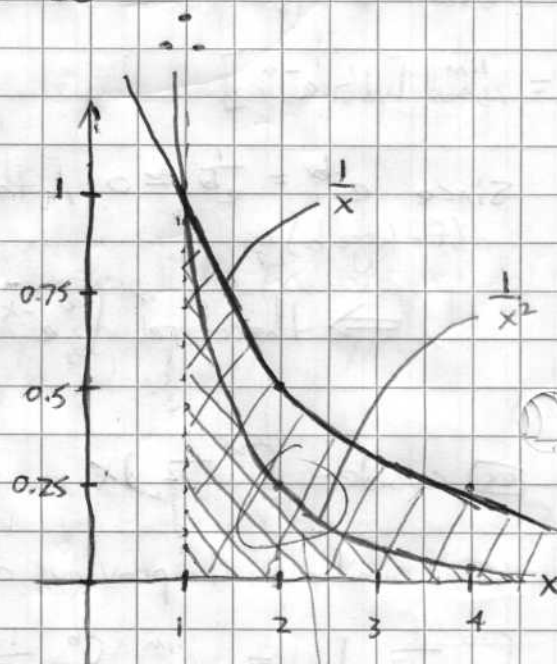
Now

$\lim_{b \rightarrow \infty} b = \infty$ and since $b = e^X$, we must have $\lim_{b \rightarrow \infty} b = \infty = e^X$.

$\rightarrow$ now for $e^X = \infty$, we must have $X = \infty$

□ this is a good example showing that just because $f(x)$ is finite on the interval and decreasing w/ x , $\int_1^{\infty} f(x) dx$ does not necessarily converge

$\Rightarrow$ convergence here ultimately depends upon whether or not $f(x)$ approaches zero quickly enough such that contributions at higher values of x are insufficiently large to contribute to the integral



the difference between the two areas is sufficient to make the distinction between whether things converge or not

□ in general, the integral $\int_1^{\infty} \frac{1}{x^p} dx$ behaves as follows:

- $p \leq 1 \rightarrow$ the integral diverges

- $p > 1 \rightarrow$ the integral converges and has a value of $\frac{1}{p-1}$

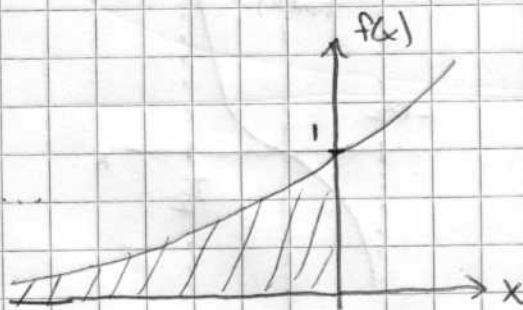
ex $\int_1^{\infty} \frac{1}{\sqrt{x}} dx \rightarrow \text{diverges}$

ex $\int_1^{\infty} \frac{1}{\sqrt{x^3}} dx \rightarrow \text{converges to } 2 \text{ (show this explicitly!)}$

ex $\int_{-\infty}^0 \frac{1}{(x-1)^2} dx$

* Intuition: try to reason out what will likely happen...

$\frac{1}{(x-1)^2} \approx \frac{1}{x^2}$ (for large $|x|$, regardless of sign)



$\rightarrow$ since $\int_1^{\infty} \frac{1}{x^2} dx$ converges, it seems likely our integral will converge

$\int_{-\infty}^0 \frac{1}{(x-1)^2} dx = \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{(x-1)^2} dx$ [simple substitution]

$= \lim_{a \rightarrow -\infty} \left. -(x-1)^{-1} \right|_a^0 = \lim_{a \rightarrow -\infty} \left[-\frac{1}{0-1} - \frac{-1}{a-1} \right]$

$= \lim_{a \rightarrow -\infty} \left(1 + \frac{1}{a-1} \right) = 1$

$\Rightarrow$ so, the integral does indeed converge and has a value of 1

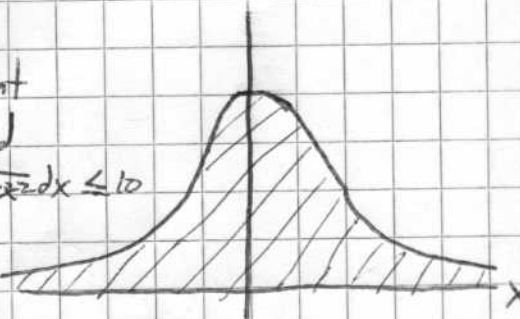
* NOTE: $\int_{-\infty}^0 \frac{1}{(x-1)^2} dx = \int_0^{\infty} e^{-x} dx = \int_1^{\infty} \frac{1}{x^2} dx = 1$

ex $\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

Intuition: for large x , $\frac{1}{1+x^2} \approx \frac{1}{x^2}$ and we know that $\int_1^{\infty} \frac{1}{x^2} dx = 1$, by symmetry $\int_{-\infty}^1 \frac{1}{x^2} dx = 1$ and $\int_{-\infty}^1 \frac{1}{x^2} dx$ is just a #, thus we expect $2 < \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx \leq 10$

$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^0 \frac{1}{1+x^2} dx + \int_0^{\infty} \frac{1}{1+x^2} dx$
 $= \lim_{a \rightarrow -\infty} \int_a^0 \frac{1}{1+x^2} dx + \lim_{b \rightarrow \infty} \int_0^b \frac{1}{1+x^2} dx =$ (cont.)

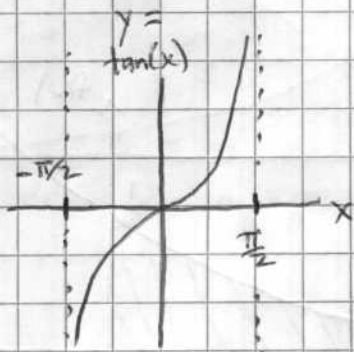
$f(x) = \frac{1}{1+x^2}$



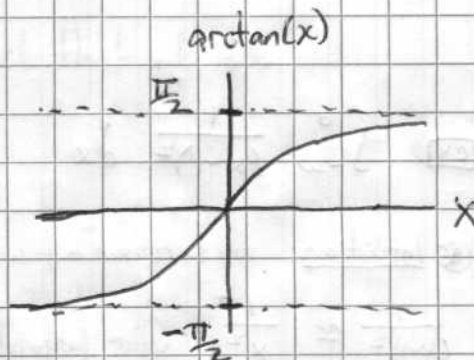
11/20/16 (cont.)

18.7 (cont.)

$$= \lim_{a \rightarrow -\infty} \arctan x \Big|_a^0 + \lim_{b \rightarrow \infty} \arctan x \Big|_0^b$$



$$\begin{aligned} y &= \tan x \\ \updownarrow \\ x &= \arctan(y) \end{aligned}$$



$$= \lim_{a \rightarrow -\infty} [\arctan(0) - \arctan(a)] + \lim_{b \rightarrow \infty} [\arctan(b) - \arctan(0)]$$

$$= 0 - (-\frac{\pi}{2}) + \frac{\pi}{2} - 0 = \boxed{\pi}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi}$$