

1/19/10

250B - Tuesday Afternoon

QUESTION

Suppose $f(x)$ is an odd function. Then is it true that $\int_{-\infty}^{\infty} f(x) dx$ converges?

→ consider a simple odd function, $f(x) = x$.
Then it seems that the two areas (since they are symmetric, but opposite in sign) cancel one another out and thus the integral is zero. Therefore it seems it does indeed converge.

or, let's consider analytically:

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$$

$$= \lim_{a \rightarrow -\infty} \int_a^0 x dx + \lim_{b \rightarrow \infty} \int_0^b x dx$$

$$= \lim_{a \rightarrow -\infty} \left. \frac{1}{2} x^2 \right|_a^0 + \lim_{b \rightarrow \infty} \left. \frac{1}{2} x^2 \right|_0^b$$

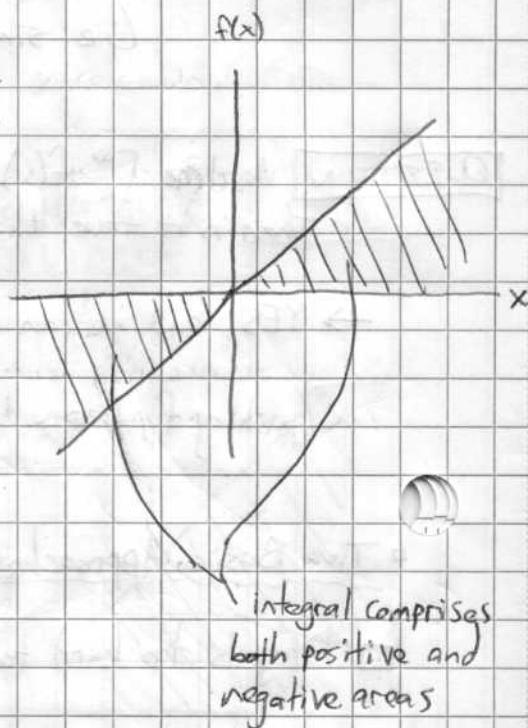
$$= \lim_{b \rightarrow \infty} \frac{1}{2} b^2 - \lim_{a \rightarrow -\infty} \frac{1}{2} a^2 = \frac{1}{2} (\lim_{b \rightarrow \infty} b^2 - b^2)$$

→ which is just $\infty^2 - \infty^2 = 0$, right?

⇒ Unfortunately, the limit makes things not so simple. Consider

$A \in [0, 1]$ (A is the set of all real #'s between 0 and 1)

$B \in [0, 2]$ (B " " " 0 and 2)



both A and B are infinitely large, but if we subtract A from B, we are not left with something that's zero (even worse, we actually still have something infinitely large!!) → PARADOX (this gets into an issue of cardinality)

thus $\infty - \infty \neq 0$, but more like $\infty - \infty = ?$

□ another way to put it (in a more general case):

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^{\infty} f(x) dx$$

now if either $\int_{-\infty}^a f(x) dx$ or $\int_a^{\infty} f(x) dx$ have a finite value, they converge (by our definition). otherwise they diverge. If one diverges, then $\int_{-\infty}^{\infty} f(x) dx$ must diverge too.

→ thus, within the context of how we define convergence, oppositely signed diverging entities don't cancel one another out

ANSWER

the statement is thus not true for a general odd function $f(x)$ (though we could find odd functions where ~~the~~ indeed $\int_{-\infty}^{\infty} f(x) dx$ would converge)

□ so it seems our paradox above could in some way stem from ambiguity in how we think about/define ∞ , motivating the subsequent question:

QUESTION II

How do we define infinity?

[for such a ubiquitous concept, one might find it surprising how rarely such a question is posed/answered!]

11/19/10 (cont.)

◦ Some early lines of thought were the notion of ∞ popped up (and caused a lot of confusion!):

◻ Zeno - paradox of Achilles racing the tortoise

◻ Pythagoras - notion of an irrational # (e.g. $\sqrt{2}$)

◻ Newton/Leibniz - notion of an infinitesimal in calculus

◻ the notion of a limit (and what it entails for such everyday things such as the continuity of a function)

◻ Hilbert - notion of the 'Grand Hotel'

→ as we have seen already, we can easily be misled by Euclid's 'commonsense' that the whole is always greater than any of its parts and fall into the trap of many a paradox!!

⇒ all this motivate a sub-question: do we even consider ∞ a number?

◻ Georg Ferdinand Ludwig Philipp Cantor (1845-1918)

- Russian/German mathematician, very controversial in his time, but considered a genius in hindsight

- pioneered the counter-intuitive (and novel) idea that that the whole is no bigger than the sum of its parts

e.g. there are just as many positive even integers as there are positive integers altogether (!)

→ this counter-intuition is precisely a defining property of infinite sets

→ however, he did go further to state there is a hierarchy of infinities

→ these ideas played a very important role in shaping early 20th century mathematics (e.g. Gödel's incompleteness theorem)