

$$\int_a^b f(x) dx \approx \sum_{i=1}^N f(x_i) \Delta x \quad [\text{Riemann sum: integral is a sum of parts}]$$

$$\int_a^b f(x) dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_i) \Delta x \quad [\text{or conversely, } \lim_{\Delta x \rightarrow 0}]$$

→ thus the integral is an infinite sum, or in some sense, a 'continuous' quantity (depending upon a and b)

Remember: a definite integral is a number

Aside: the notion of something being infinite is not trivial and sometimes fraught w/ apparent contradictions. For example, consider the following two sets of #'s:

$$A \in [1, 2] \quad \text{and} \quad B \in [1, 3]$$

A contains all real number between (and including) 1 and 2 and thus is infinite. But the set B contains everything in A and more. Thus B is bigger than A . But how can something be bigger than something else that is already infinitely big?

→ one potential way out of this is to be careful about how you define whether or not something is infinite, or put another, what it means for something to be indeed infinite (we'll come back to this a bit later....)

QUESTION

what if one (or both) of the bounds was infinite? or the integrand becomes infinite on the interval? (we call these IMPROPER INTEGRALS)

⇒ what value does such an integral take? Is it finite or infinite?

11/15/10 (cont.)

◦ some examples of improper integrals:

$$\int_{-1}^1 \frac{1}{x^2} dx$$

(integrand blows up on the interval)

$$\int_{-\infty}^{\infty} x dx, \int_{-\infty}^4 f(x) dx$$

(bounds are infinite)

$$\int_1^{\infty} \frac{1}{x} dx$$

(infinite bound and integral blows up on the interval!)

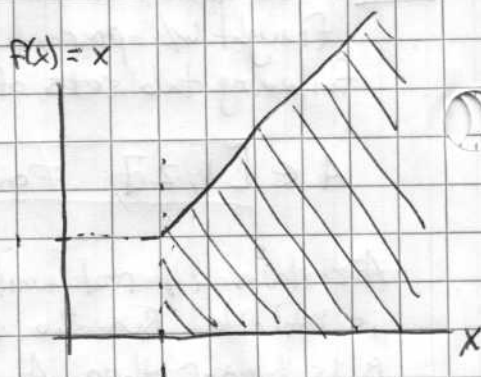
◦ most basically, one of two things can happen:

- if the integral has a finite value, we say it **CONVERGES**
- " " infinite value, " **DIVERGES**

→ let's look at some simple examples...

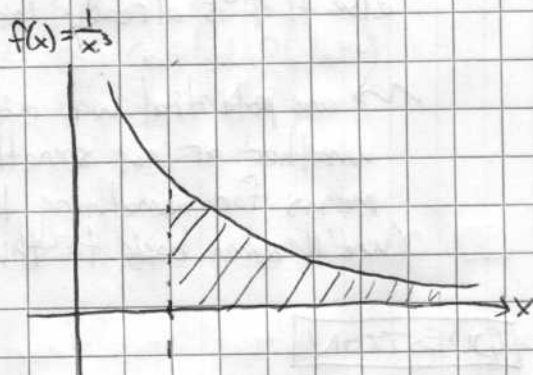
ex $\int_1^{\infty} x dx$

→ clearly this diverges
(consider the 'area' under the curve concept of the integral)



ex $\int_1^{\infty} \frac{1}{x^3} dx$

→ this will converge
(as x increases, eventually additional contributions become negligible)



ex $\int_1^{\infty} \frac{1}{x} dx$

→ this will actually diverge

→ how come $\int_1^{\infty} \frac{1}{x^3} dx$ converges then but $\int_1^{\infty} \frac{1}{x} dx$ does not?

$\Rightarrow$ For $\int_a^\infty f(x) dx$ to have a chance to converge, we must have $f(x) \rightarrow 0$ for increasing x ; however, such a condition is not sufficient to guarantee convergence!

ex $\int_{-\infty}^\infty \sin x dx \rightarrow$ though the value of the integral is bounded between -1 and 1 , it will never take on a finite value (i.e. $\sin(\infty) = ?$) and therefore we say this integral diverges

QUESTION Assume $\int_a^\infty f(x) dx$ converges, $f(x) > 0$ for $x > a$ and $b > a$. Then is it true that $\int_b^\infty f(x) dx$ converges?

$\rightarrow$ YES, and we can even go one step further, saying: $\int_b^\infty f(x) dx < \int_a^\infty f(x) dx$ (pushing further, does $f(x)$ need to be continuous for this to hold?)

Two Basic Approaches to Dealing w/ Improper Integrals

1] Integrals too hard to solve $\rightarrow$ comparison (HH 7.8; covered next week)

2] Solvable Integrals $\rightarrow$ take the limit (HH 7.7)

assume $f(x)$ is continuous on the interval

$$\bullet \int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx = \lim_{b \rightarrow \infty} [F(b) - F(a)]$$

$$\bullet \int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx = \lim_{a \rightarrow -\infty} [F(b) - F(a)]$$

$$\bullet \int_{-\infty}^\infty f(x) dx = \int_{-\infty}^a f(x) dx + \int_a^\infty f(x) dx$$

$\rightarrow$ this last one allows us to handle the more general case where $f(x)$ is not continuous on the interval

QUESTION Suppose $f(x)$ is an odd function. Then is it true that $\int_{-\infty}^\infty f(x) dx$ converges?