

Math 250A (Fall 2009) - Quiz 6

SOLUTIONS

1. (5 points) Find a unique solution to the following differential equation given the specified initial condition

$$\frac{dy}{dx} = \frac{1 + e^{4x}}{\sqrt{4x + e^{4x}}} \quad y(0) = 5$$

To find a solution to this differential equation, we must use the method of separation of variables.

$$\int dy = \int \frac{1 + e^{4x}}{\sqrt{4x + e^{4x}}} dx$$

We can now see that the integral on the right can be solved with substitution. We let $u = 4x + e^{4x}$. Therefore, $du = 4 + 4e^{4x} dx$. We can now rewrite and solve our integral.

$$y = \int \frac{\frac{1}{4}(4 + 4e^{4x})}{\sqrt{4x + e^{4x}}} dx = \frac{1}{4} \int \frac{du}{\sqrt{u}} = \frac{1}{4} [2\sqrt{u}] + C = \frac{1}{2} \sqrt{4x + e^{4x}} + C$$

Next, our initial condition is applied to our general solution.

$$y(0) = \frac{1}{2} \sqrt{4(0) + e^0} + C = \frac{1}{2} \sqrt{1} + C = \frac{1}{2} + C = 5$$

$$\Rightarrow C = 5 - \frac{1}{2} = \frac{9}{2}$$

$$\Rightarrow y(x) = \frac{1}{2} \sqrt{4x + e^{4x}} + \frac{9}{2}$$

2. (5 points) Evaluate the following integral. **Hint:** $\int \sin^2 \theta d\theta = \frac{1}{2} [-\sin \theta \cos \theta + \theta] + C$

$$\int \frac{3x^2}{\sqrt{25 - x^2}} dx$$

Let $x = 5 \sin(\theta)$ and $dx = 5 \cos(\theta) d\theta$.

$$\begin{aligned} \int \frac{3x^2}{\sqrt{25 - x^2}} dx &= \int \frac{3(5 \sin(\theta))^2 (5 \cos(\theta))}{\sqrt{25 - (5 \sin(\theta))^2}} d\theta = \int \frac{5(75 \sin^2(\theta) \cos(\theta))}{5\sqrt{1 - \sin^2(\theta)}} d\theta \\ &= \int \frac{75 \sin^2(\theta) \cos(\theta)}{\sqrt{\cos^2(\theta)}} d\theta = 75 \int \sin^2(\theta) d\theta = \frac{75}{2} [-\sin \theta \cos \theta + \theta] + C \end{aligned}$$

From our substitution, we know $\sin(\theta) = \frac{x}{5}$ and $\theta = \arcsin(\frac{x}{5})$. If we consider a right triangle with an angle θ , we know the length of the opposite side of the triangle is x and the length of the hypotenuse is 5. Therefore, we can use the Pythagorean theorem to find the length of the adjacent side, l , and the value of $\cos(\theta)$.

$$l^2 + x^2 = 25 \Rightarrow l^2 = 25 - x^2 \Rightarrow l = \sqrt{25 - x^2}$$

$$\Rightarrow \cos(\theta) = \frac{l}{5} = \frac{\sqrt{25 - x^2}}{5}$$

Now, we can substitute our values of θ , $\sin(\theta)$, and $\cos(\theta)$ back into our solution.

$$\begin{aligned} \frac{75}{2} [-\sin \theta \cos \theta + \theta] + C &= \frac{75}{2} \left[-\frac{x}{5} \frac{\sqrt{25 - x^2}}{5} + \arcsin\left(\frac{x}{5}\right) \right] + C \\ &= -\frac{3}{2} x \sqrt{25 - x^2} + \frac{75}{2} \arcsin\left(\frac{x}{5}\right) + C \end{aligned}$$