

Math 250A (Fall 2009) - Quiz 5

SOLUTIONS

1. (5 points) A dam has a rectangular base 1500 meters long and 160 meters wide. Its cross section is shown below. By slicing horizontally, set up and evaluate a definite integral giving the volume of material used to build this dam.

Let us first consider how to find the area of the cross section. We know that the area can be found with the simple equation $A_{cs} = \frac{1}{2}h(a+b)$. We can use this to check our integral.

There are many ways to approach this problem, so we will consider a series of horizontal slices with a width of w and a height of $\triangle h$. Since w decreases linearly as h increases, let us assume that w has the form $w = mh + c$ where m and c are constants. Since we know the values of w at $h = 0$ and $h = 150$ m, we can find the values of m and c .

$$\begin{aligned}w(0) &= m(0) + c = c = 160 \\ \Rightarrow w(150) &= m(150) + 160 = 10 \Rightarrow m = \frac{10 - 160}{150} = -1 \\ \Rightarrow w &= -h + 160\end{aligned}$$

Now, we can integrate w from 0 to 150 to find the area of the cross section.

$$\int_0^{150} -h + 160 \, dh = \left[-\frac{1}{2}h^2 + 160h\right]_0^{150} = 24,000 - 11,250 = 12,750$$

To check the result of our integral, let's use the known formula for the area of a trapezoid.

$$A_{cs} = \frac{1}{2}(150 \text{ m})(10 \text{ m} + 160 \text{ m}) = 12,750 \text{ m}^2$$

Now, to find the volume of the dam, we integrate the area of the cross section over the length of the dam.

$$\begin{aligned}V_{dam} &= \int_0^{1500} A_{cs} \, dx = A_{cs}[x]_0^{1500} \\ &= (1500)(12750) = 19,125,000 \text{ m}^3\end{aligned}$$

2. (5 points) Find the coefficients A , B , and C , of the partial fraction decomposition of the following function

$$f(x) = \frac{x}{(x-3)(x+7)(x-8)} = \frac{A}{x-3} + \frac{B}{x+7} + \frac{C}{x-8}$$

To start, we must combine the fractions over a common denominator and set the numerator equal to x .

$$\begin{aligned}
& A(x+7)(x-8) + B(x-3)(x-8) + C(x-3)(x+7) \\
&= A(x^2 - x - 56) + B(x^2 - 11x + 24) + C(x^2 + 4x - 21) \\
&= (A+B+C)x^2 + (-A-11B+4C)x + (-56A+24B-21C) = x
\end{aligned}$$

Since both the coefficient of x^2 and the constant are zero, we can set up a system of equations to find the values of A , B , and C .

$$\begin{aligned}
A + B + C &= 0 \\
-A - 11B + 4C &= 1 \\
-56A + 24B - 21C &= 0
\end{aligned}$$

After reducing the system of equations, we find the following:

$$\begin{aligned}
A + B + C &= 0 \\
B - \frac{1}{2}C &= -\frac{1}{10} \\
C &= \frac{8}{75}
\end{aligned}$$

Since the value of C is known, we can now find the values of B and A .

$$A = -\frac{3}{50}, B = -\frac{7}{150}, C = \frac{8}{75}$$

An alternative method is to consider our equation

$$A(x+7)(x-8) + B(x-3)(x-8) + C(x-3)(x+7) = x$$

The equation should hold for all values of x . Therefore, let us choose values of x that greatly reduce our equation. In other words, we will let $x = 3, -7, 8$.

$$\begin{aligned}
3 &= A(10)(-5) + B(0)(-5) + C(0)(10) = -50A \\
-7 &= A(0)(-15) + B(-10)(-15) + C(-10)(0) = 150B \\
8 &= A(15)(0) + B(5)(0) + C(5)(15) = 75C
\end{aligned}$$

We still obtain the same values for A , B , and C .

Extra Credit (3 points) The population (N) of a city is modeled by the (logistic) equation given below, where r is assumed to be constant. Solve this differential equation and obtain a function that describes the city's population for any time t .

$$\frac{dN}{dt} = rN(1 - N)$$

To solve this differential equation, we use the technique of separation of variables. We rewrite our equation as follows.

$$\frac{dN}{N(1 - N)} = r dt$$

We can now see that we will need to use partial fraction decomposition to integrate the left side of the equation.

$$\int \frac{dN}{N(1-N)} = \int \frac{A}{N} + \frac{B}{1-N} dN$$

$$\Rightarrow (-A + B)N + A = 1$$

$$\Rightarrow A = 1 \text{ and } -A + B = 0 \Rightarrow A = B = 1$$

$$\Rightarrow \int \frac{dN}{N(1-N)} = \int \frac{1}{N} + \frac{1}{1-N} dN$$

$$= \ln |N| - \ln |1-N| = \ln \left| \frac{N}{1-N} \right| = rt + C_1$$

$$\Rightarrow \frac{N}{1-N} = C_2 e^{rt} \text{ where } C_2 = e^{C_1}$$

$$\Rightarrow N = (1-N)C_2 e^{rt} \Rightarrow N + C_2 N e^{rt} = N(1 + C_2 e^{rt}) = C_2 e^{rt}$$

$$\Rightarrow N(t) = \frac{C_2 e^{rt}}{1 + C_2 e^{rt}}$$