

Math 250A (Fall 2009) - Quiz 11

SOLUTIONS

1. (5 points) Solve the following differential equation:

$$\frac{d\psi}{dt} + \frac{4}{t}\psi + \pi = 4t$$

First, let us rewrite our differential equation.

$$\frac{d\psi}{dt} + \frac{4}{t}\psi = 4t - \pi$$

In this form, we can easily see $p(t) = 4/t$ and $q(t) = 4t - \pi$. Therefore, our solution is in the form:

$$\psi(t) = e^{-\int p(t) dt} \int q(t) e^{\int p(t) dt} dt$$

We will now solve the integral of $p(t)$.

$$\int \frac{4}{t} dt = 4 \ln |t| + C \quad \Rightarrow \quad e^{\int p(t) dt} = A e^{4 \ln |t|} = t^4$$

assuming $C = 0$ (or conversely, $A = 1$). Similarly, $e^{-\int p(t) dt} = t^{-4}$.

$$\psi(t) = t^{-4} \int (4t - \pi) t^4 dt = t^{-4} \int 4t^5 - \pi t^4 dt = t^{-4} (4/6 t^6 - \pi/5 t^5 + \kappa)$$

$$\Rightarrow \quad \psi(t) = \frac{2}{3} t^2 - \frac{\pi}{5} t + \frac{\kappa}{t^4}$$

where κ is a constant dependent upon the initial condition.

2. (5 points) Solve the initial value problem:

$$\frac{dy}{dx} = (y^2 + 9) \tan(x) \quad y(0) = 3\sqrt{3}$$

[Hint: $\int \frac{1}{k^2 + \theta^2} d\theta = \frac{1}{k} \arctan(\theta/k) + C$]

This differential equation is separable, so we will use the method of separation of variables to solve it.

$$\int \frac{dy}{y^2 + 9} = \int \tan(x) dx$$

The right side of the equation is solved by using the substitution $u = \cos(x)$.

$$\int \tan(x) dx = \int \frac{\sin(x)}{\cos(x)} dx = - \int \frac{du}{u} = - \ln |u| + C = - \ln |\cos(x)| + C$$

In order to solve the left side of our separated equation, we will use the given substitution letting $\theta = y$ and $k = 3$.

$$\int \frac{dy}{y^2 + 9} = \frac{1}{3} \arctan(y/3)$$

Now, we combine our results.

$$\frac{1}{3} \arctan(y/3) = -\ln|\cos(x)| + C \quad \Rightarrow \quad \arctan(y/3) = -3\ln|\cos(x)| + C$$

We can now apply our initial condition to solve for our constant.

$$\arctan(3\sqrt{3}/3) = \arctan(\sqrt{3}) = \frac{\pi}{3} = -3\ln|\cos(0)| + C = -3\ln(1) + C = C$$

$$\Rightarrow \quad C = \frac{\pi}{3}$$

Now, we will solve for y .

$$\arctan(y/3) = -3\ln|\cos(x)| + \pi/3 \quad \Rightarrow \quad y/3 = \tan(-3\ln|\cos(x)| + \pi/3)$$

$$y(x) = 3 \tan(-3\ln|\cos(x)| + \pi/3)$$