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## Sequences (see H.H. 9.1)

• question: what do the following #s have in common (e.g. any pattern)?

0, 1, 4, 9, 16, 25, 36, ----

Answer: each one is an integer squared  $[0^2, 1^2, 2^2, 3^2, 4^2, \dots]$

• what about: 0, 1, 3, 7, 15, 31, 63, 127, ----

Answer: each one (after the <sup>first</sup> ~~second~~ entry) is  $\times 2$  the previous entry plus 1  
 $[0, 2(0)+1, 2(1)+1, 2(3)+1, 2(7)+1, 2(15)+1, \dots]$

• what about:  $\frac{1}{2}, -\frac{1}{4}, \frac{1}{6}, -\frac{1}{8}, \frac{1}{10}, \dots$

Answer: getting smaller and alternating sign, but perhaps we need some sort of formalism to better express things

Aside: Note that we essentially have a list of #s and that in some sense, there is a 'starting point', but no 'end point'

$\leadsto$  rewrite as follows:

|                          |               |                |               |                |                |         |
|--------------------------|---------------|----------------|---------------|----------------|----------------|---------|
| <u>entries</u> ( $s_n$ ) | $\frac{1}{2}$ | $-\frac{1}{4}$ | $\frac{1}{6}$ | $-\frac{1}{8}$ | $\frac{1}{10}$ | $\dots$ |
| <u>index</u> ( $n$ )     | 1             | 2              | 3             | 4              | 5              | $\dots$ |

a positive integer  
(though it need not  
start @ 1!)

$\leadsto$  so how does each entry depend upon  $n$ ?

$$s_n = 2n^{-1} \cdot (-1)^{n+1} = -1^{n+1} \cdot \frac{1}{2n} \quad [\text{NOTE: also could have } s_n = -1^{n-1} \cdot \frac{1}{2n}]$$

$\Rightarrow$  so we can write our 'sequence' of #s as  $s_n = \frac{-1^{n+1}}{2n}$   
starting w/  $n=1$  (such that each value explicitly depends upon its 'index')

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 conversely, we could write the 'sequence' 0, 1, 3, 7, 15, 31, 63, ... as

$$S_{n+1} = 2 \cdot S_n + 1 \quad \text{where } S_0 = 0 \quad \text{OR} \quad S_n = 2^n - 1 \quad \text{starting @ } n=0$$

[we call this type of definition recursive]

Definition: A sequence has the form  $S_n = f(n)$  and can be written in order of increasing  $n$  (eg.  $S_0, S_1, S_2, S_3, \dots$ ). A sequence can also be defined recursively as  $S_n = f(S_{n-1})$  or  $S_{n+1} = f(S_n)$

ex find an expression for  $S_n$  for the following sequence:  
 1, 3, 5, 7, 9, ...

[determine the pattern!]  $\rightarrow$   
 (some trial + error can help if you get stuck)

functional formula  $\rightarrow$

$$S_n = \begin{cases} 2n+1 & \text{starting at } n=0 \\ S_{n-1}+2 & \text{start} \end{cases}$$

recursive formula

ex similar as above for  $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \dots$

$$S_n = \frac{n+1}{2n+3}$$

for  $n=0$  as the starting point

since  $n:$

|                    |               |               |               |                |     |
|--------------------|---------------|---------------|---------------|----------------|-----|
| 0                  | 1             | 2             | 3             | 4              | ... |
| seq: $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{3}{7}$ | $\frac{4}{9}$ | $\frac{5}{11}$ | ... |

NOTE: Can you come up w/ a recursive definition?  
 (also see next pg.)

NOTE: Suppose we started at  $n=1$  instead of  $n=0$ . then we'd have  
 $S_n = \frac{n}{2n+1}$   
 (which is also correct!)

$\Rightarrow$  so it's important to also explicitly state what your starting value for  $n$  is

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ex compute  $a_n$  for  $n = 1, 2, 3, 4, 5$  for  $a_{n+1} = \frac{1}{4}a_n + \frac{3}{4}$  where  $a_0 = 2$

$$a_1 = \frac{1}{4}(2) + \frac{3}{4} = \frac{5}{4}$$

$$a_2 = \frac{1}{4}\left(\frac{5}{4}\right) + \frac{3}{4} = \frac{17}{16}$$

$$a_3 = \frac{1}{4}\left(\frac{17}{16}\right) + \frac{3}{4} = \frac{65}{64}$$

$$a_4 = \frac{1}{4}\left(\frac{65}{64}\right) + \frac{3}{4} = \frac{257}{256}$$

$$a_5 = \frac{1}{4}\left(\frac{257}{256}\right) + \frac{3}{4} = \frac{1025}{1024}$$

→ note that there is a clear pattern here:

$$s_n = \frac{4^n + 1}{4^n}$$

→ so a recursively defined sequence can also be defined in 'functional form'

$$s_{n+1} = f(s_n) \iff s_n = f(n)$$

∴

what if we also had  $s_{n+1} = 3s_n + 2n$  where  $s_0 = 3$ ? this would also be a sequence, one with the form

$$s_{n+1} = f(n, s_n)$$

→ does this look familiar (re. ODEs)?

Think Euler's method!

$$y_{n+1} = y_n + \Delta t \cdot f(t_n, y_n)$$

→ thus numerically solving an ODE over a discretized interval (i.e.  $\Delta t$ ) parallels 'iterating' a recursive relation



$$y(t + \Delta t) = y(t) + f(t, y(t)) \cdot \Delta t$$

→ so numerically solving an ODE is in essence similar to computing a recursively defined sequence! we can even go one step further, considering <sup>solving</sup> a 'discretized' ODE essentially as a recursively defined sequence:

$$\frac{dx}{dt} = Ax(1-x)$$

[Logistic Equation]



(parallels between the two)

$$x_{n+1} = r x_n (1 - x_n)$$

[Quadratic 'Map', or Quadratic Recurrence Relation]

→ just like we numerically solve an ODE via Euler's method by moving ahead in time steps (which we discretize!), we can essentially do the same thing by iterating the map