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## Further Examples of Bernoulli ODEs

**EX** Consider  $x' = x - x^2 \sin t$

→ Bernoulli ODE where  $n=2$

let  $u = x^{1-2} = \frac{1}{x}$  (assumes  $x \neq 0$ )

$$\text{then } \frac{du}{dt} = -\frac{1}{x^2} \cdot x' = -\frac{1}{x^2} [x - x^2 \sin t] = -\frac{1}{x} + \sin t \\ = -u + \sin t$$

$$\rightarrow u' + u = \sin t \quad [p(t)=1, q(t)=\sin t]$$

$$\int p(t) dt = \int dt = t \quad (\text{let arb. const.} = 0)$$

$$e^{\int p(t) dt} = e^t$$

$$\Rightarrow u(t) = e^{-t} \int \sin t \cdot e^t dt + C e^{-t}$$

→ end up w/ an integral we need to solve using IBPs:

$$\int e^t \sin t dt = -\cos t e^t + \int e^t \cos t dt$$

$$= -e^t \cos t + e^t \sin t - \int e^t \sin t dt$$

$$\begin{array}{l} a=e^t \rightarrow a'=e^t \\ b'=\sin t \rightarrow b=-\cos t \end{array}$$

$$\begin{array}{l} c=e^t \rightarrow c'=e^t \\ d'=\cos t \rightarrow d=\sin t \end{array}$$

$$\rightarrow \int e^t \sin t dt = \frac{1}{2} e^t [\sin t - \cos t] + k$$

back to our solution:

$$u(t) = e^{-t} \left[ \frac{1}{2} e^t (\sin t - \cos t) + k \right] + C e^{-t}$$

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$$= \frac{1}{2} \sin t - \frac{1}{2} \cos t + \frac{k'}{e^t} = u(t) = \frac{1}{x}$$

$$\Rightarrow x(t) = \frac{1}{\frac{1}{2} \sin t - \frac{1}{2} \cos t + \frac{k'}{e^t}} = \frac{2}{\sin t - \cos t + \tilde{C} e^{-t}}$$

Note: in deriving this solution, we implicitly assumed  $x \neq 0$  (so to make the subst.  $u = \frac{1}{x}$ ), thus to write the most general solution, we have:

$$x(t) = \begin{cases} \frac{2}{\sin t - \cos t + \tilde{C} e^{-t}} & x \neq 0 \text{ (}\tilde{C} \text{ is an arb. const.)} \\ 0 & x = 0 \end{cases}$$

...

ex  $\phi' = a\phi - e^x \phi^3$  (where  $\phi' = \frac{d\phi}{dx}$  and  $a$  is a const.)

$\rightarrow$  Bernoulli w/  $n=3$ . Let  $u = \phi^{-2} \rightarrow u' = -2a u + 2e^x$

can solve through to obtain the general solution

$$\phi(x) = \begin{cases} \left( \pm \left( c e^{-2at} + \frac{2}{1+2a} e^t \right) \right)^{-1/2} & a \neq -1/2 \\ \pm (c e^t + 2t e^t)^{-1/2} & a = 1/2 \\ 0 & x = 0 \end{cases}$$

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ex solve following initial value problem;

$$y' + 2y = 4t, \quad y(0) = 5 \quad (y' = \frac{dy}{dt})$$

$$p(t) = 2t \rightarrow \int 2t dt = t^2 \rightarrow e^{\int p(t) dt} = e^{t^2}$$

$$\rightarrow y(t) = e^{-t^2} \int 4te^{t^2} dt + Ce^{-t^2}$$

$$= e^{-t^2} [2e^{t^2} + k] + Ce^{-t^2} = 2 + \tilde{C}e^{-t^2}$$

$$y(0) = 5 = 2 + \tilde{C} \rightarrow \tilde{C} = 3 \Rightarrow \boxed{y(t) = 2 + 3e^{-t^2}}$$

→ what does this solution look like?

- so is  $y=2$  an equilibrium solution?
- if so, is it stable/unstable?
- phase line diagram?

