

Example

$$\text{ex } y' + \ln(x)y = \tan(x)$$

(presumably $x > 0$)
 $\rightarrow p(x) = \ln x, q(x) = \tan x$

$$\int p(x) dx = \int \ln x dx = x \ln x - x \quad (+C) \quad [\text{using IBP}]$$

$\downarrow$ let $C=0$

$$\rightarrow e^{\int p(x) dx} = \exp[x \ln x - x] = \cancel{x^x} \cdot x^x \cdot e^{-x} = \frac{x^x}{e^x}$$

[can solve through]

$$y(x) = \frac{e^x}{x^x} \int \tan(x) \cdot \frac{x^x}{e^x} dx + C \frac{e^x}{x^x}$$

hard to integrate! $\therefore$

NOTE: Jakob Bernoulli was a famous Swiss mathematician who first described these types of ODEs way back in 1695

Bernoulli Eqns (see L.L. 5.3)

defined as eqns. of the form: $y' + p(x)y = q(x)y^n \quad (n \neq 1)$

Idea: can we somehow make a substitution so to turn this into a linear ODE?

let $u \equiv y^{1-n}$ (where n is a const. stemming from the ODE)

$$\text{then } \frac{du}{dx} = u' = \frac{d}{dx}(y^{1-n}) = (1-n)y^{-n} \cdot \frac{dy}{dx} = (1-n)y^{-n} \cdot y'$$

$$= (1-n)y^{-n} [q(x)y^n - p(x)y]$$

$$= (1-n) [q(x) - p(x)y \cdot y^{-n}]$$

$$= (1-n) [q(x) - p(x)y^{1-n}] = (1-n) [q(x) - p(x)u] = u'$$

we already know this from the ODE!!

$\Rightarrow$ So this is just a linear first order ODE for $u(x)$
 [which in turn leads back to $y(x)$!]

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(cont.)

NOTE: nonlinear, but fits our description
of a 'Bernoulli eqn'

ex $y' = ay - y^3$

arb. const.

rewrite as: $y' - ay = -y^3$

$[p(x) = -a, q(x) = -1, n = 3]$

let $u = y^{1-n} = y^{-2} = \frac{1}{y^2}$

[NOTE: this implicitly assumes $y \neq 0$,
see further comments on this below]

$$u' = -\frac{2}{y^3} y' = -\frac{2}{y^3} [ay - y^3] = -2ay^{-2} + 2$$
$$= -2au + 2$$

$\rightarrow$ so we have $u' + 2au = 2$ [which we know how to solve']

$\tilde{p}(x) = 2a, \tilde{q}(x) = 2$

$\int \tilde{p}(x) dx = \int 2a dx = 2ax$ (letting the const. of int. = 0)

so $e^{\int \tilde{p}(x) dx} = e^{2ax}$ and $e^{-\int \tilde{p}(x) dx} = e^{-2ax}$

$\rightarrow u(x) = e^{-2ax} \int 2e^{2ax} dx + Ce^{-2ax}$

$$= e^{-2ax} \left[\frac{2}{2a} e^{2ax} + k \right] + Ce^{-2ax}$$

$$= \frac{1}{a} + \frac{\tilde{C}}{e^{2ax}} = u = \frac{1}{y^2}$$

then $y^2 = \frac{1}{\frac{1}{a} + \frac{\tilde{C}}{e^{2ax}}} = \frac{a}{1 + \tilde{C}ae^{-2ax}}$

then $y(x) = \pm \sqrt{\frac{a}{1 + \tilde{C}ae^{-2ax}}}$

• assume we have the I.C. $y(0) = y_0$, then
($y_0 \neq 0$)

$$y_0^2 = \frac{a}{1+\tilde{C}a}, \quad 1+\tilde{C}a = \frac{a}{y_0^2} \rightarrow \tilde{C} = \frac{1}{y_0^2} - \frac{1}{a} = \frac{a-y_0^2}{ay_0^2}$$

[NOTE: also, we simply have $\tilde{C}a = \frac{a}{y_0^2} - 1$]

$$\text{then } y(x) = \pm \sqrt{\frac{a}{1+\tilde{C}ae^{-2ax}}} = \pm \sqrt{\frac{a}{1+(\frac{a}{y_0^2}-1)e^{-2ax}}} = \pm \sqrt{\frac{ay_0^2}{y_0^2+(a-y_0^2)e^{-2ax}}}$$

→ so it looks like we got an answer, but can we go further? Let's take a step back and look @ our ODE in a slightly different light:

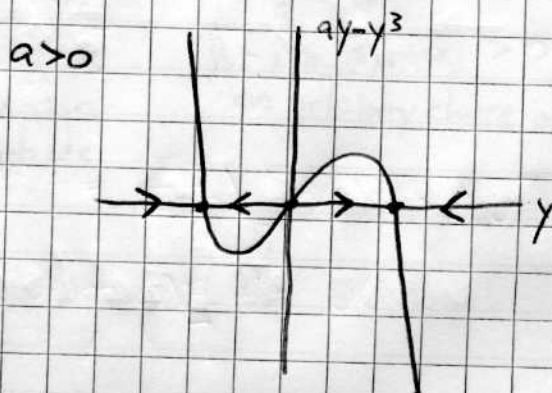
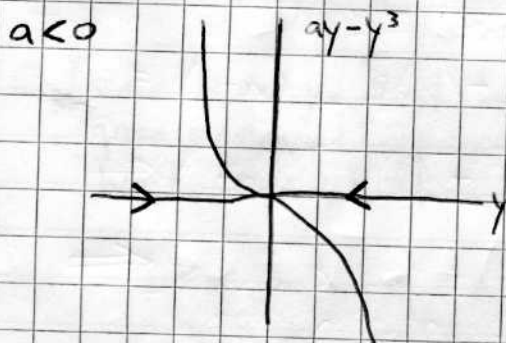
$$y' = ay - y^3$$

• equilibria: $y' = 0 = y(a - y^2) \rightarrow y_{eq} = 0, \pm\sqrt{a}$

NOTE: we 'lost' our $y=0$ equilibrium solution in our derived solution above because we explicitly assumed $y \neq 0$ when we made the substitution $u = y^{-2}$; also note that if $a \leq 0$, we only have one equilibrium solution ($y=0$), but if $a > 0$ then we have three equilibria

→ this change in the behavior of ODE's solutions w/ respect to the const. a is called a bifurcation

• stability: we can determine the stability by sketching a phase-line diagram



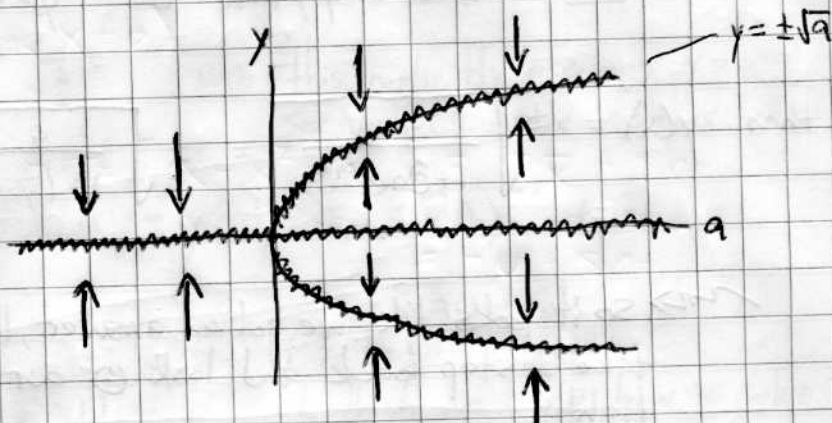
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(cont.)

■ bifurcation diagram [for $y' = ay - y^3$ example cont.]

→ from our phase-line diagram, we can make a sketch as to how the qualitative behavior of solutions change w/ respect to changes in a

~~~~~ - equil.  $y$ -values  
[arrows indicate stability]



⇒ clearly something major happens at  $a=0$

- we go from having one equilibrium solution to three
- the origin goes from stable to unstable

→ this is an example of a supercritical pitchfork bifurcation

## ■ Existence + Uniqueness (E+U)

$$y' = ay - y^3 = f(y)$$

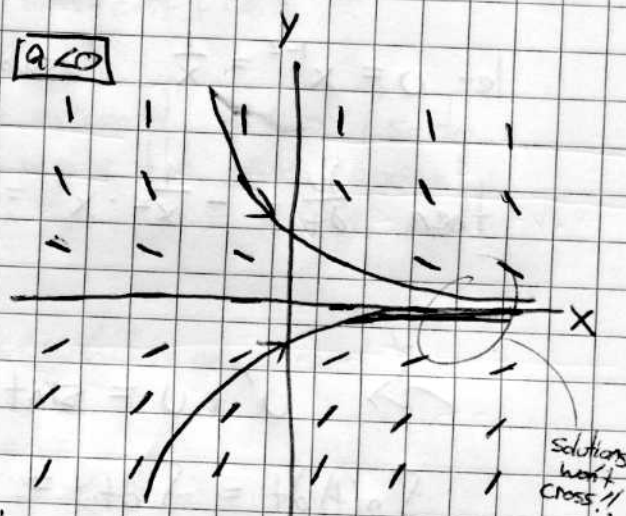
$$\frac{df}{dy} = a - 3y^2$$

→ both  $f(y)$  and  $\frac{df}{dy}$  are continuous for all values of  $x$  and  $y$ , thus the E+U Theorem indicates that all points in the plane will have a unique solution passing through them (independent of the value of  $a$ )

## Slope Field

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→ taking together everything we've done up to this point, it should be pretty straight-forward to sketch the slope field for our ODE



[Autonomous ODE; behavior similar along horizontal axis]

QUESTION: what would slope field look like for  $a = 0$  case?

⇒ back to our analytical solution:

$$\begin{aligned}
 y &= \pm \sqrt{\frac{ay_0^2}{y_0^2 + (a - y_0^2)e^{-2ax}}} = \pm \sqrt{y_0^2} \sqrt{\frac{a}{y_0^2 + (a - y_0^2)e^{-2ax}}} \\
 &= \pm y_0 \sqrt{\frac{a}{y_0^2 + (a - y_0^2)e^{-2ax}}} \\
 &= y_0 \sqrt{\quad}
 \end{aligned}$$

[Since  $y$  and  $y_0$  must have the same sign given existence + uniqueness, i.e. Solutions won't cross  $y = 0$ ]

QUESTION:

what guarantees that  $\frac{a}{y_0^2 + (a - y_0^2)e^{-2ax}} > 0$  for an arbitrary choice of  $y_0, a$ ?

∴