

11/16/09

Solving Linear 1st order ODEs

ex Solve $x' = -3x + 2e^{-t}$ (where $x' = \frac{dx}{dt}$)

w/ initial condition $x(0) = 5$

• rewrite: $x' + 3x = 2e^{-t} \rightsquigarrow p(x) = 3, q(x) = 2e^{-t}$

• Solve for the 'integrating factor':

$$\int p(t) dt = \int 3 dt = 3t + C$$

$$\text{so } e^{\int p(t) dt} = e^{3t+C} = Ae^{3t} \rightsquigarrow e^{3t} \text{ (letting } A=1\text{)}$$

• multiply both sides of ODE by $e^{\int p(t) dt}$

$$e^{\int p(t) dt} (x' + p(t)x) = e^{3t} (x' + 3x) = \frac{d}{dt} (e^{3t} \cdot x)$$

$$\text{original right-hand side} \rightsquigarrow = 2e^{-t} \cdot e^{3t} = 2e^{2t}$$

• now integrate both sides (after separating variables):

$$\int d(e^{3t}x) = e^{3t}x = \int 2e^{2t} dt = e^{2t} + C$$

• solve for $x(t)$: $\rightarrow x(t) = Ce^{-3t} + e^{-t}$

• plug in I.C.: $x(0) = 5 = Ce^0 + e^0 = C + 1 \rightarrow C = 4$

$$\rightarrow \boxed{x(t) = 4e^{-3t} + e^{-t}}$$

$\rightsquigarrow$ can plug back into ODE to verify this indeed is a solution $\therefore$

11/16/09 (cont.)

ex $xx' = 2x + 1$

→ rewrite: $x' = 2 + \frac{1}{x} \Rightarrow$ not linear (but separable)

→ could solve via separation of variables and integration [but may end up w/ something where it is hard to explicitly solve for $x(t)$]

ex solve the following ODE: $x' = (-\frac{1}{t})x + \sin t$

$$x' + \frac{x}{t} = \sin t \rightarrow p(t) = \frac{1}{t}, q(t) = \sin t$$

$$\int p(t) dt = \int \frac{1}{t} dt = \ln|t| + C$$

$$e^{\int p(t) dt} = e^{\ln|t| + C} = A|t| = t \quad (\text{letting } A=1)$$

$$e^{-\int p(t) dt} = \frac{1}{t}$$

$$\text{then } x(t) = e^{-\int p(t) dt} \int q(t) e^{\int p(t) dt} dt + \frac{C}{e^{\int p(t) dt}}$$

$$= \frac{1}{t} \int t \sin t dt + \frac{C}{t}$$

$$= \frac{1}{t} [-t \cos t + \sin t + C'] + \frac{C}{t} \quad (\text{use IBP})$$

$$\boxed{= \frac{\sin t}{t} - \cos t + \frac{C}{t} = x(t)}$$

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RL-circuit

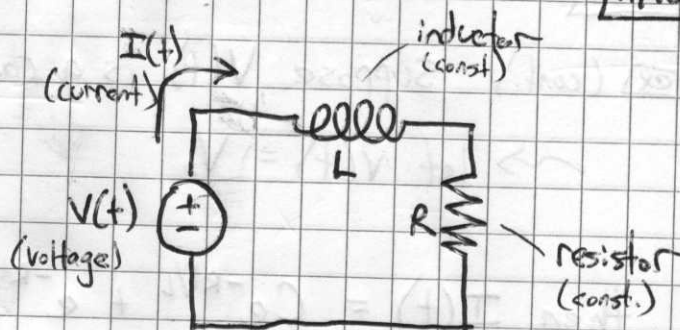
use Kirchhoff's Law

$$V(t) = L \frac{dI(t)}{dt} + RI(t)$$

driving voltage
(could be DC or AC)

voltage drop across inductor

voltage drop across resistor



Aside: depending upon the form of $V(t)$, the mechanical analog is a mass moving across a rough surface

⇒ linear, first-order ODE for $I(t)$!!

$$\frac{dI}{dt} + \frac{R}{L} I = V(t) \cdot \frac{1}{L}$$

$$p(t) = \frac{R}{L}$$

$$q(t) = \frac{V(t)}{L}$$

$$\int p(t) dt = \int \frac{R}{L} dt = \frac{R}{L} t$$

(let const. of integration be zero)

$$\rightarrow e^{\int p(t) dt} = e^{Rt/L}$$

so we can rewrite our ODE as: $e^{Rt/L} \frac{dI}{dt} + \frac{R}{L} e^{Rt/L} I = \frac{V(t)}{L} e^{Rt/L}$

$$\text{i.e. } \frac{d}{dt} (I \cdot e^{Rt/L}) = \frac{V(t)}{L} e^{Rt/L}$$

⇒ which we can now integrate

$$I e^{Rt/L} = \int_0^t \frac{V(s)}{L} e^{Rs/L} ds + C \quad (\text{integrating from } t=0 \text{ to some time } t)$$

$$\rightarrow I(t) = C e^{-Rt/L} + e^{-Rt/L} \int_0^t \frac{V(s)}{L} e^{Rs/L} ds$$

so depending upon the form of $V(t)$, we still have an integral to solve

Cont.

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ex (cont.) Suppose $V(t)$ is a const. (e.g. 9V, such as you would get when connecting a battery)
 $\leadsto$ let $V(t) = V_0$

$$\text{then } I(t) = C e^{-Rt/L} + e^{-Rt/L} \int_0^t \frac{V_0}{L} e^{Rs/L} ds$$

$$= C e^{-Rt/L} + e^{-Rt/L} \cdot \frac{V_0}{L} \left[\frac{L}{R} e^{Rs/L} \right]_0^t$$

$$= C e^{-Rt/L} + \frac{V_0}{R} e^{-Rt/L} (e^{Rt/L} - 1)$$

$$= \tilde{C} e^{-Rt/L} + \frac{V_0}{R}$$

$\Rightarrow$ so an initial condition will determine $\tilde{C}$, but regardless, the current will asymptotically approach the value V_0/R (the rate at which it does so being determined by the ratio R/L)

NOTE: if $L=0$, this simply reduces to Ohm's Law ($V=IR$) as expected