

11/9/09

250 Lecture (L.L. Ch.5.1)

Goal: Develop a means to handle linear ODEs: $y' + p(x)y = q(x)$

→ motivating example: consider $xy' + y = \cos(x)$

note that $xy' + y = \frac{d}{dx}(xy) = (xy)' = \cos(x)$

and that

$$\int (xy)' dx = xy \Rightarrow xy = \int \cos x dx = \sin x + C$$

$$\rightarrow y(x) = \frac{1}{x} [\sin(x) + C]$$

- so in this case, we could see that the left-hand side of the ODE was an exact derivative, but that will not usually be so clear!!

Idea: In order to so $y' + p(x)y = q(x)$, can we make a clever choice for re-expressing $y(x)$ such that the equation becomes separable (and thus readily solvable via integration)?

→ will build off the idea above, using a couple ideas developed below

$$\square y'f(x) + yg(x) = \frac{d}{dx}(yf) = y'f + yf' \quad [\text{i.e. } g(x) = f'(x)]$$

$$\square h(x)[y' + p(x)y] = \frac{d}{dx}[yh(x)]$$

$$\text{where } h'(x) = p(x)h(x) = \frac{dh}{dx} \quad (\text{this is separable!})$$

$$\int \frac{dh}{h} = \int p(x) dx \rightarrow \ln|h| = \int p(x) dx + C$$

$$\leadsto h(x) = \pm e^c \exp\left(\int p(x) dx\right) = A e^{\int p(x) dx}$$

for simplicity, let $A=1$, then we have:

$$e^{\int p(x) dx} [y' + p(x)y] = \frac{d}{dx} [y \cdot e^{\int p(x) dx}] \quad \left[\text{since } \frac{d}{dx} \left(\int p(x) dx \right) = p(x) \right]$$

③ let's go back to our linear ODE: $y' + p(x)y = q(x)$

• multiply both sides by $h(x) = e^{\int p(x) dx}$

$$e^{\int p(x) dx} [y' + p(x)y] = q(x) \cdot e^{\int p(x) dx}$$

left hand side is an exact derivative!

$$e^{\int p(x) dx} [y' + p(x)y] = \frac{d}{dx} [y \cdot e^{\int p(x) dx}]$$

• so our ODE becomes: $\frac{d}{dx} [y \cdot e^{\int p(x) dx}] = q(x) e^{\int p(x) dx}$

• we can integrate/solve this:

$$\int d[y e^{\int p(x) dx}] = y \cdot e^{\int p(x) dx} = \int q(x) e^{\int p(x) dx} dx + C$$

const. of integration $\swarrow$

• solve for y :
$$y(x) = \frac{\int q(x) e^{\int p(x) dx} dx + C}{e^{\int p(x) dx}}$$

$$= e^{-\int p(x) dx} \cdot \int q(x) e^{\int p(x) dx} dx + C \cdot e^{-\int p(x) dx}$$

11/9/09 (cont.)

see pgs. 190-191

• NOTE: notational aside; the book (LL) uses $e^{\int p(x) dx} \equiv \mu(x)$ which it calls the integrating factor

Summary

[A] if we have a linear, first order ODE (which pop up in a large # of practical applications), we can rewrite it as

$$y' + p(x)y = q(x)$$

[Z] we can write a general solution to this ODE as:

$$y(x) = \frac{1}{\mu(x)} \int q(x) \mu(x) dx + \frac{C}{\mu(x)}$$

$$\text{where } \mu(x) = \exp\left[\int p(x) dx\right]$$

(see 'Comments about Linear Diff. Eqns. on L.L. pgs. 191-192; e.g., why did we pick $A=1$ when solving for $h(x)$ on the previous pg.?)

NOTE: this is also called the variation of constants formula, which is sometimes stated in the following theorem:

Theorem: if $p(x)$ and $q(x)$ are continuous on an interval $a < x < b$ containing x_0 , then a unique solution to the initial value problem

$$y' = p(x)y + q(x) \quad y(x_0) = y_0$$

is given by

$$y(x) = y_0 e^{P(x)} + e^{P(x)} \int_{x_0}^x e^{-P(u)} q(u) du$$

$$\text{where } P(x) = \int_{x_0}^x p(s) ds \quad \therefore$$