

Solve $y' = \frac{x^2 + xy + y^2}{xy} = g(x, y) \rightarrow h\left(\frac{y}{x}\right)$

make substit. $z \equiv \frac{y}{x}$

$$y' = g\left(\frac{y}{x}\right) \rightarrow z'x + z = f(z)$$

when to do? $g(x, y) \rightarrow g(x, \alpha x)$

$$\frac{x^2 + xy + y^2}{xy} = \frac{x^2 + \alpha x^2 + \alpha^2 x^2}{\alpha x^2} = \frac{1 + \alpha + \alpha^2}{\alpha}$$

$$\frac{x}{y} + 1 + \frac{y}{x} = \frac{1}{z} + 1 + z$$

$$y' = \frac{x^2 + xy + y^2}{xy} \Rightarrow z'x + z = \frac{1}{z} + 1 + z \quad (\text{where } z = \frac{y}{x})$$

solve for z' : $z' = \frac{1}{x} \left(1 + \frac{1}{z}\right)$

integrate $\int \frac{1}{1 + \frac{1}{z}} dz = \int \frac{1}{x} dx$

$$\int \frac{z}{z+1} dz = \ln|x| + C$$

$$\int \frac{z}{z+1} dz = \int \frac{z+1-1}{z+1} dz = \int \left(1 - \frac{1}{z+1}\right) dz$$

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$$= z - \ln|z+1| = \ln|x| + C \quad \text{(simplify first, then exponentiate)}$$

$$z = \ln|x| + \ln|z+1| + C = \ln|x(z+1)| + C$$

$$e^z = A|x(z+1)| = \pm A x(z+1) \quad \text{(remember } z = \frac{y}{x} \text{)}$$

$$Ae^{y/x} = x\left(\frac{y}{x} + 1\right) = x + y \quad \text{[can't solve for } y(x) \text{?]}$$

Summary: $Ae^{y/x} = x\left(1 + \frac{y}{x}\right)$ is a solution to $y' = \frac{x^2 + xy + y^2}{xy}$

Check our answer by plugging back into the ODE

$$\frac{d}{dx} \left(Ae^{y/x} \right) = Ae^{y/x} \frac{d}{dx} \left(\frac{y}{x} \right) = -Ae^{y/x} \frac{y}{x^2}$$

wrong! need to use the chain rule!

$$= Ae^{y/x} \left(\frac{y'}{x} - \frac{y}{x^2} \right) = \frac{d}{dx} (x+y) = 1 + y'$$

$$= \frac{d}{dx} \left[x \left(1 + \frac{y}{x} \right) \right] = \left(1 + \frac{y}{x} \right) + x \left(\frac{y'}{x} - \frac{y}{x^2} \right)$$

$$= (x+y) \left(\frac{y'}{x} - \frac{y}{x^2} \right) \quad (?)$$

$$y' \left(1 - \frac{x+y}{x}\right) = -1 - \frac{y}{x^2} (x+y)$$

↓ let's rewind a bit and start over

$$Ae^{\frac{y}{x}} = x+y$$

we already know what this is!!

$$\frac{d}{dx}(x+y) = 1+y' = \frac{d}{dx}(Ae^{\frac{y}{x}}) = Ae^{\frac{y}{x}} \left(\frac{y'}{x} - \frac{y}{x^2}\right)$$

$$= (x+y) \left(\frac{y'}{x} - \frac{y}{x^2}\right) = 1+y'$$

$$y' - \frac{y}{x} + \frac{xy'}{x} - \frac{y^2}{x^2} = 1+y'$$

$$y' \left(1 - 1 - \frac{y}{x}\right) = -\frac{y}{x} y' = -\frac{y}{x} - \frac{y^2}{x^2} - 1$$

$$y' = \frac{x}{y} \left(\frac{y}{x} + \frac{y^2}{x^2} + 1\right) = 1 + \frac{y}{x} + \frac{x}{y}$$

$$= \frac{xy + y^2 + x^2}{xy} = y'$$

checks out!

∴

ex $2xy \frac{dy}{dx} = 4x^2 + 3y^2$

- does it have 'homogeneous coefficients'?
- if so, can we use our developed methods to solve?
- remember to check your answers afterwards to see if ~~the~~ it does indeed 'solve' the ODE