

11/4/09

PROBLEM SOLVING RND II. (see online notes)

11/6/09

250A Lecture (L.L. 4.2)

ODEs w/ Homogeneous Coefficients

Goal: solve $y' = \frac{x^2 + xy + y^2}{xy}$

→ highly nonlinear, not separable

⇒ how to approach? can we simplify?

Aside: Consider an eqn. of the form $y' = g\left(\frac{y}{x}\right)$

→ so g does depend upon x and y , but in a way such that it is the ratio of y to x that determines its form (we call this an ODE w/ HOMOGENEOUS COEFFICIENTS)

let $z \equiv \frac{y}{x}$, then $y = zx$

then $y' = \frac{dy}{dx} = \frac{d}{dx}(xz) = z + x \frac{dz}{dx} = z + xz'$

back to the ODE: $y' = g\left(\frac{y}{x}\right) = g(z) = z + xz'$

→ $z' = \frac{g(z) - z}{x}$ which is actually separable!!

So now we need to solve (upon separating variables):

$$\int \frac{dz}{g(z) - z} = \int \frac{1}{x} dx$$

→ hopefully from here, we can solve for $z(x)$ and thereby $y(x) = x \cdot z(x)$!

→ back to our problem: $y' = \frac{x^2 + xy + y^2}{xy}$

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QUESTION: Is this ODE homogeneous [i.e. can we write it as $y' = g(\frac{y}{x})$]?

→ to determine this, set $y = \alpha x$ (where α is a const.):

$$\frac{x^2 + xy + y^2}{xy} = \frac{x^2 + \alpha x^2 + \alpha^2 x^2}{x^2 \alpha} = \frac{1 + \alpha + \alpha^2}{\alpha^2}$$

⇒ if $g(x, \alpha x)$ does not depend upon x , then g is homogeneous of degree 0 and we can apply the method of homogeneous coefficients

letting $z = \frac{y}{x}$, → $y' = \frac{x^2 + xy + y^2}{xy} = \frac{x}{y} + 1 + \frac{y}{x}$

$$= z'x + z = \frac{1}{z} + 1 + z$$

→ $z' = \frac{1}{x} \left(\frac{1}{z} + 1 \right)$ → which we know how to solve!!

$$\int \frac{1}{x} dx = \int \frac{1}{\frac{1}{z} + 1} dz = \int \frac{z}{1+z} dz = \int \frac{1+z-1}{1+z} dz$$

$$= \int \left(1 - \frac{1}{1+z} \right) dz = z - \ln|1+z| = \ln|x| + C$$

$$z - C = \ln|x| + \ln|1+z| = \ln|x(1+z)| \quad (\text{exponentiate both sides})$$

$$|x(1+z)| = e^{z-C} = e^z e^{-C} \rightarrow x(1+z) = Ae^z \quad (A \text{ is a const.})$$

now $z = \frac{y}{x} \Rightarrow \boxed{x\left(1 + \frac{y}{x}\right) = Ae^{y/x}}$

→ can't readily solve for $y(x)$, but we can at least check our solution in order to verify whether it satisfies the ODE