

Euler's Method

Goal: develop means to numerically solve ODEs to obtain a solution curve.

Starting Point: $x' = \frac{dx}{dt} = f(t, x)$ w/ I.C. $x(t_0) = x_0$

→ we want to now 'approximate' the solution $x = x(t)$ of $x' = f(x, t)$ at $t = t_1 > t_0$, the basic idea being to integrate the ODE

this approx is a key conceptual piece of the puzzle

$$x' = f(x, t) \approx \frac{x(t + \Delta t) - x(t)}{\Delta t} \quad (\text{by definition of the derivative})$$

$$\rightarrow x(t + \Delta t) = x(t) + f(x, t) \Delta t$$

$\uparrow$ x at some future time $\uparrow$ t at some previous time $\uparrow$ rate of change of x (the ODE tells us this!)

time step over which to see how x has changed

▣ let's put another way: Fundamental Theorem of Calc.

(being more precise)

if $x' = f(t, x)$, then $x(t_1) - x(t_0) = \int_{t_0}^{t_1} f(t, x(t)) dt$

since $x(t_0) = x_0$, then $x(t_1) = x_0 + \int_{t_0}^{t_1} f(t, x) dt$

→ now we may or may not know how to solve the integral $\int f(t, x) dt$ (if we do, then we can analytically find a solution!), but let's take the approach here of approx. it via a Riemann sum:

Left-hand Rule: $\int_{t_0}^{t_1} f(t, x) dt \approx \underbrace{(t_1 - t_0)}_{\text{width of rect.}} \underbrace{f(t_0, x(t_0))}_{\text{height of rect. on 'left'}}$

11/2/09 (cont.)

let $\Delta t = t_1 - t_0$ [assume $t_1 > t_0$, $x(t_1) = x_1$, $x(t_0) = x_0$]

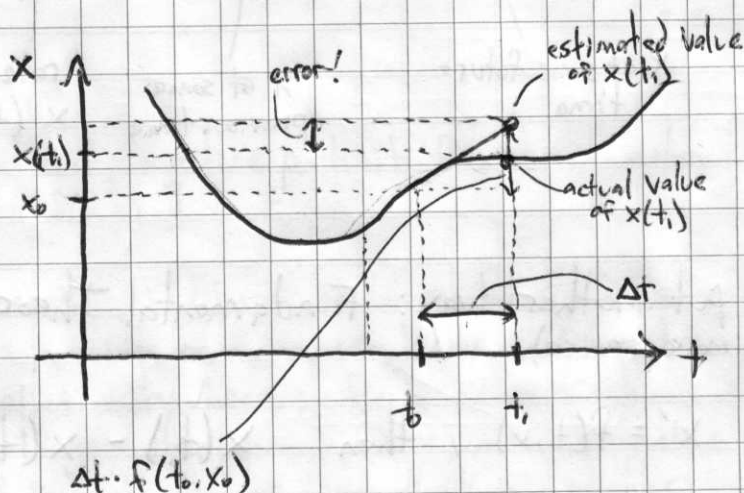
$\rightarrow x_1 \approx x_0 + \Delta t \cdot f(t_0, x_0)$ [like we had already concluded!]

□ note that one could 'keep going': $x_2 = x_1 + \Delta t \cdot f(t_1, x_1)$
(where $\Delta t = t_2 - t_1$)

$\rightarrow$ and we could keep iterating along for successive values of t and effectively trace out the solution curve

ERROR

□ note that we only get an approx. solution, and thus there will be some error associated w/ it (easiest to illustrate this graphically):



NOTE: if you get a bit confused, think about the units of the various quantities

$\rightarrow$ the problem is that the slope will likely change over the interval $t \in [t_0, t_1]$, thus we will end up over/under-estimating how much $x(t)$ really changed over that interval

Accuracy

error is the diff. between the exact value (i.e. the analytic function) that is the solution to the ODE and the estimated value

the smaller the step-size, the smaller the error (but at the sacrifice of having to do more computations!)

if the sign of $f(t, x)$ changes a lot, sometimes errors can cancel one another out!!

put another way: the error is proportional (approx.) to the # of steps over the interval (e.g., increasing from 2 steps to 10 will reduce the assoc. error by a factor of 5)

could also use a better choice to estimate the integral (i.e. TRAP or MID over LEFT); this leads to the modified/improved Euler method and Runge-Kutta

estimate $y(1)$, where $y(x)$ is the solution to the ODE $\frac{dy}{dx} = x^3 - y^3$ w/ the I.C. $y(0) = 0$ and step-size $\Delta x = \frac{1}{5}$

NOTE: ODE is very nonlinear and hard to solve analytically!!